Techniques of Integration
積分技巧

§9-1

$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$

§9-2 Integral Tables（積分表）

$\int f(u)du = F(u) + C$

$\int f(g(x))g'(x)dx = F(g(x)) + C$

Basic Forms

1、$\int udv = uv - \int vdu$

2、$\int u^2 du = \frac{u^{n+1}}{n+1} + C$ ，$n \neq -1$

3、$\int \frac{du}{u} = \ln|u| + C$

4、$\int e^u du = e^u + C$

5、$\int a^u du = \frac{a^u}{\ln a} + C$

6、$\int \sin udu = -\cos u + C$

7、$\int \cos udu = \sin u + C$

8、$\int \sec^2 udu = \tan u + C$

9、$\int \csc^2 udu = -\cot u + C$

10、$\int \sec u \tan udu = \sec u + C$

11、$\int \csc u \cot udu = -\csc u + C$

12、$\int \tan udu = \ln|\sec u| + C$

13、$\int \cot udu = \ln|\sin u| + C$

14、$\int \sec udu = \ln|\sec u + \tan u| + C$

15、$\int \csc udu = \ln|\csc u - \cot u| + C$

16、$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17、$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18、$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19、$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20、$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

§9-3 三角積分

半角公式：$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

<例題 1>：$\int \sin^2 3x dx = ?$

<解答>：$\int \sin^2 3x dx = \int \frac{1}{2}(1-\cos 6x)dx$

$$=\frac{1}{2}(x-\frac{1}{6}\sin 6x)+C=\frac{1}{2}x-\frac{1}{12}\sin 6x+C$$

平方關係：$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$1+\cot^2\theta=\csc^2\theta$

<例題 2>：$\int \cot^2 3x dx = ?$

<解答>：$\int \cot^2 3x dx = \int\left(\csc^2 3x-1\right)dx$

$$=\int\left(\csc^2 u-1\right)\left(\frac{1}{3}du\right) \qquad (\ u=3x\ )$$

$$=\frac{1}{3}\left(-\cot u-u\right)+C$$

$$=-\frac{1}{3}\cot 3x-x+C$$

$u=\sin x$

$du=\cos x dx$

$$\int \sin^3 x\cos x dx=\int u^3 du$$

$$=\frac{1}{4}u^4+C=\frac{1}{4}\sin^4 x+C$$

$u=\cos x$

$du=-\sin x dx$

以上可以用在$\int \sin^m x\cos^n x dx$這類式子中

$$D_x(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$u(x)v'(x)=D_x\left[u(x)v(x)\right]-v(x)u'(x)$$

$$\int u(x)v'(x)dx=u(x)v(x)-\int v(x)u'(x)dx$$

$$du=u'(x)dx$$

$$dv=v'(x)dx$$

➲ $\boxed{\int udv=uv-\int vdu}$

§9-5 理性函數與分數

$$R(x)=\frac{P(x)}{Q(x)}$$

$$R(x)=\frac{P(x)}{Q(x)}=p(x)+F_1(x)+F_2(x)+...+F_k(x)$$

<例題 1>： $\int\frac{x^3-1}{x^3+x}dx=?$

<解答>： $\int\frac{x^3-1}{x^3+x}dx=\int\left(1-\frac{1}{x}+\frac{x-1}{x^2+1}\right)dx$

$$=\int\left(1-\frac{1}{x}+\frac{x}{x^2+1}-\frac{1}{x^2+1}\right)dx$$

$$=x-\ln|x|+\frac{1}{2}\ln(x^2+1)-\tan^{-1}x+C$$

<例題 2>： $\int\frac{x^3+x^2+x-1}{x^2+2x+2}dx=?$

<解答>：

$$\begin{array}{r|l} & x-1 \\ x^2+2x+2 & x^3+\ x^2+\ x+1 \\ & \underline{x^3+2x^2+2x} \\ & -x^2-\ x-1 \\ & \underline{-x^2-2x-2} \\ & x+1 \end{array}$$

$$\frac{x^3+x^2+x-1}{x^2+2x+2}=(x-1)+\frac{x+1}{x^2+2x+2}$$

$$\int\frac{x^3+x^2+x-1}{x^2+2x+2}dx=\int\left(x-1+\frac{x+1}{x^2+2x+2}\right)dx$$

$$=\frac{1}{2}x^2-x+\frac{1}{2}\ln\left(x^2+2x+2\right)+C$$

§9-6 Trigonometric Substitution

If the integral involves	Then substitute	And use the identity
a^2-u^2	$u=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
a^2+u^2	$u=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
u^2-a^2	$u=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

<例題 1>： $\int\frac{x^3}{\sqrt{1-x^2}}=?,\text{where}\,|x|<1$

<解答>： $x=\sin\theta$

$dx=\cos\theta d\theta$

$$\int\frac{x^3}{\sqrt{1-x^2}}=\int\frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$

$$=\int\sin^3\theta d\theta$$

$$=\int\sin\theta(1-\cos^2\theta)d\theta$$

$$=\frac{1}{3}\cos^3\theta-\cos\theta+C$$

$\because\ \cos\theta=\left(1-\sin^2\theta\right)^{\frac{1}{2}}=\left(1-x^2\right)^{\frac{1}{2}}$

$\therefore\ \int\frac{x^3}{\sqrt{1-x^2}}=\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}-\sqrt{1-x^2}+C$ ...#

If the integral involves	Then substitute	And use the identity
a^2-u^2	$u=a\tanh\theta$	$1-\tanh^2\theta=\text{sech}^2\theta$
a^2+u^2	$u=a\sinh\theta$	$1+\sinh^2\theta=\cosh^2\theta$
u^2-a^2	$u=a\cosh\theta$	$\cosh^2\theta-1=\sinh^2\theta$