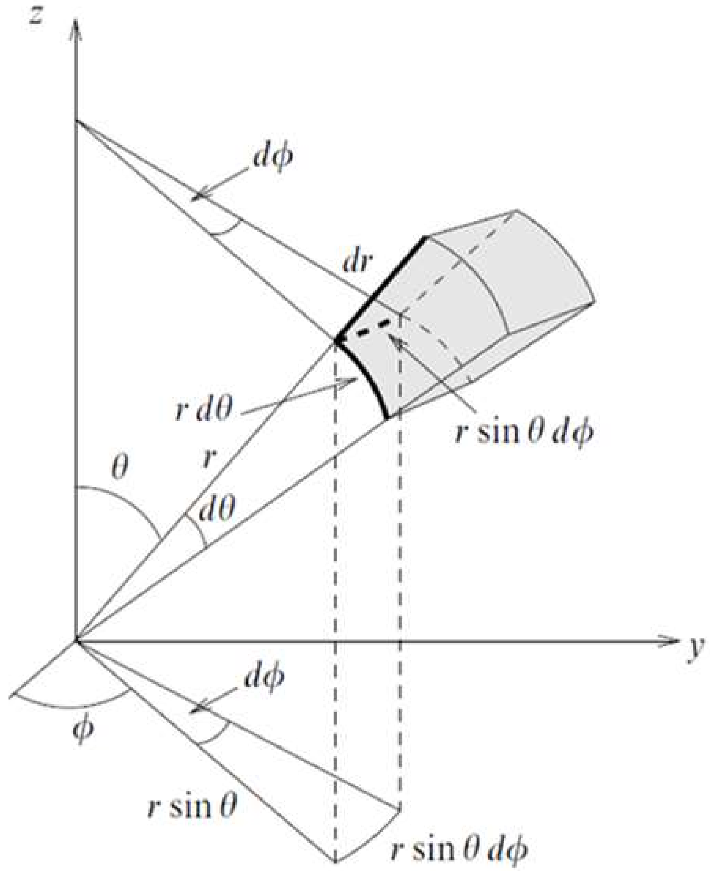
Problem 1

三維空間中卡氏座標的$P(1,2,3)$點，請計算P在圓柱坐標系統系統中的3個座標變數的數值？請計算在P點的三個圓柱坐標系統的基底向量：$(\hat{e}_\rho,\ \hat{e}_\phi,\ \hat{e}_z)$。 (01

Problem 1 (A)

$P(1,2,3)$



$x = \rho\cos\phi$, $\rho = \sqrt{x^2+y^2} = \sqrt{5}$

$y = \rho\sin\phi$, $\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.4^\circ$

$= 1.107\,\text{rad}$

$z = 3$

$$\hat{e}_\rho = \frac{(1,2,0)}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}}(1,2,0)$$

$$\hat{e}_\phi = \frac{1}{\sqrt{5}}(-2,1,0)$$

$$\hat{e}_z = (0,0,1)$$

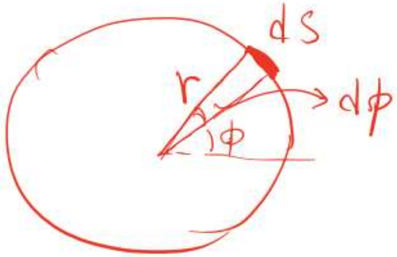
$$\vec{e}_1 = \frac{\partial\vec{r}}{\partial u_1},\quad \vec{e}_2 = \frac{\partial\vec{r}}{\partial u_2},\quad \vec{e}_3 = \frac{\partial\vec{r}}{\partial u_3}$$

$$\hat{e}_1 = \frac{1}{h_1}\vec{e}_1,\quad \hat{e}_2 = \frac{1}{h_2}\vec{e}_2,\quad \hat{e}_3 = \frac{1}{h_3}\vec{e}_3$$

請用一維、二維和三維積分推導出圓週長、圓面積和球體積的公式。假設圓和球的半徑為r。 (01小題)

Problem 1 (B)

圓週長：



$$ds = r\,d\phi$$

$$C = \int ds = \int_0^{2\pi} r\,d\phi$$

$$= r\int_0^{2\pi} d\phi$$

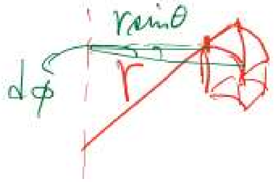
$$= 2\pi r$$

圓面積：

$$dA = \frac{1}{2} r^2 d\phi$$

$$A = \int dA = \frac{1}{2} r^2 \int d\phi = \pi r^2$$

球体積：



$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

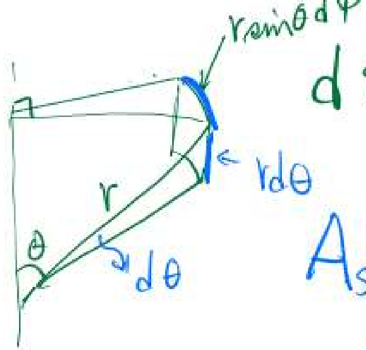
$$= (r^2 dr)(\sin\theta\,d\theta)(d\phi)$$

$$V = \int dV = \int_0^r r^2 dr \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \left(\frac{1}{3} r^3\right)\left(-\cos\theta\Big|_0^{\pi}\right)(2\pi)$$

$$= \frac{4}{3}\pi r^3$$

球表面積：



$$ds = (r\sin\theta\,d\phi)(r\,d\theta)$$

$$A_s = \int ds$$

$$= \int_0^{2\pi} r\sin\theta\,d\phi \int_0^{\pi} r\,d\theta$$

$$= r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta$$

$$= r^2 (2\pi)(2) = 4\pi r^2$$

Problem 2

- 三維空間的球座標系統是由三個座標變數$(r,\ \theta,\ \phi)$所定義的座標系統，請寫下這三個坐標變數與卡氏座標的(x, y, z)變數之間的轉換關係。
- 在球坐標系統中也有三個基底向量，這三個基底向量與卡式座標的基底向量不同，基底向量的方向會隨著坐標點位置的不同而改變。請你敘述這三個基底向量$(\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi)$是如何定義和推導，並且根據這個定義推導出這3個基底向量。
- 在推導這三個基底向量的過程中，其實我們就已經得到了球座標系的3個h因子(廣義曲線正交坐標系的h因子)，請你寫下球座標系統的3個因子。

(a)
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ \theta = \cos^{-1}\dfrac{z}{r} \end{cases}$$

$$\hat{e}_r = \frac{\vec{r}}{r}, \qquad \frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left[r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} \right] = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$|\vec{e}_\theta| = r, \quad \hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

(c)
$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right| = 1$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r, \quad h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta$$

for spherical polars $\quad h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta.$

Problem 3

承上題，

- 有了這些基本的認識請你來計算下面定義在球坐標系統中的純量函數

$$\Phi(r,\theta,\phi) = r\cos\theta + r\sin\theta\cos\phi$$

的梯度向量和Laplace算符作用後所得到的結果。

- 請針對下面定義在球坐標系當中的向量

$$\vec{a} = (r\sin\phi\hat{e}_r + rhate_\theta)$$

的散度和旋度。

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The equations for del operators in a general curvilinear coodinate is given below:

$$\mathbf{h}_1 = \frac{\partial \mathbf{r}}{\partial u_1};\ \mathbf{h}_2 = \frac{\partial \mathbf{r}}{\partial u_2};\ \mathbf{h}_3 = \frac{\partial \mathbf{r}}{\partial u_3}.$$

$$h_1 = |\mathbf{h}_1|;\ h_2 = |\mathbf{h}_2|;\ h_3 = |\mathbf{h}_3|$$

$$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{u_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{u_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{u_3}$$

$$\nabla\cdot\vec{a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3a_1) + \frac{\partial}{\partial u_2}(h_3h_1a_2) + \frac{\partial}{\partial u_3}(h_1h_2a_3)\right]$$

$$\nabla\times\vec{a} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

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Problem 2

(a) $\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$ $\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}\frac{y}{x} \\ \theta = \cos^{-1}\frac{z}{r} \end{cases}$

(b) $\hat{e}_r = \frac{\vec{r}}{r}$, $\vec{e}_\theta = \frac{\partial\vec{r}}{\partial\theta} = \frac{\partial}{\partial\theta}\left[r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}\right]$

$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j} + \cos\theta\,\hat{k}$

$= r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,k$

$|\vec{e}_\theta| = r$, $\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$\vec{e}_\phi = \frac{\partial\vec{r}}{\partial\phi} = r\sin\theta(-\sin\phi)\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} - 0\,\hat{k}$

$|\vec{e}_\phi| = r\sin\theta$, $\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

(c)

$h_r = \left|\frac{\partial\vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right|$

$= 1$

$h_\theta = \left|\frac{\partial\vec{r}}{\partial\theta}\right| = r$, $h_\phi = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = r\sin\theta$

$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\hat{e}_\varphi$

$\frac{\partial\Phi}{\partial r} = \cos\theta + \sin\theta\cos\phi$

$\frac{\partial\Phi}{\partial\theta} = -r\sin\theta + r\cos\theta\cos\phi$

$\frac{\partial\Phi}{\partial\varphi} = -r\sin\theta\sin\phi$

$= (\cos\theta + \sin\theta\cos\phi)\hat{e}_r + (-\sin\theta + \cos\theta\cos\phi)\hat{e}_\theta - \sin\phi\,\hat{e}_\varphi$

$$\nabla^2\phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\phi}\right)\right]$$

$$\vec{a} = (r\sin\phi\,\hat{e_r} + r\,\hat{e_\theta})$$

$$\nabla\cdot\vec{a} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cdot r\sin\phi)\right.$$

$$\left.+\frac{\partial}{\partial\theta}(r\sin\theta\cdot r) + 0\right]$$

$$= \frac{1}{r^2\sin\theta}\left[3r^2\sin\theta\sin\phi + r^2\cos\theta\right]$$

$$= 3\sin\phi + \cot\theta$$

$$\nabla\times\vec{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r\sin\phi & r^2 & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[0\,\hat{e_r} + (r^2\cos\phi)\hat{e_\theta} + (3r^2\sin\theta - r^2\cos\theta\sin\phi)\hat{e_\phi}\right]$$

$$= \frac{\cos\phi}{\sin\theta}\,\hat{e_\theta} + (3 - \cot\theta\sin\phi)\hat{e_\phi}$$

The equations for del operators in a general curvilinear coodinate is given below:

$$\mathbf{h}_1 = \frac{\partial \mathbf{r}}{\partial u_1};\ \mathbf{h}_2 = \frac{\partial \mathbf{r}}{\partial u_2};\ \mathbf{h}_3 = \frac{\partial \mathbf{r}}{\partial u_3}.$$

$$h_1 = |\mathbf{h}_1|;\ h_2 = |\mathbf{h}_2|;\ h_3 = |\mathbf{h}_3|$$

$$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\hat{u_1} + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\hat{u_2} + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\hat{u_3}$$

$$\nabla \cdot \vec{a} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 a_1) + \frac{\partial}{\partial u_2}(h_3 h_1 a_2) + \frac{\partial}{\partial u_3}(h_1 h_2 a_3)\right]$$

$$\nabla \times \vec{a} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial u_3}\right)\right]$$

(c)

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right| = 1$$

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r,\quad h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta$$

$$\vec{a} = (r\sin\phi\,\hat{e}_r + r\,\hat{e}_\theta)$$

$$\nabla\cdot\vec{a} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cdot r\sin\phi) + \frac{\partial}{\partial\theta}(r\sin\theta\cdot r) + 0\right]$$

$$= \frac{1}{r^2\sin\theta}\left[3r^2\sin\theta\sin\phi + r^2\cos\theta\right]$$

$$= 3\sin\phi + \cot\theta$$

$$\nabla\times\vec{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r\sin\phi & r^2 & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[0\,\hat{e}_r + (r^2\cos\phi)\hat{e}_\theta + (3r^2\sin\theta - r^2\cos\theta\sin\phi)\hat{e}_\phi\right]$$

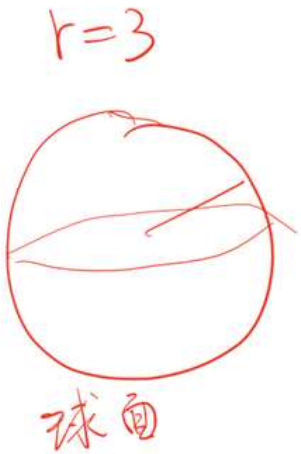
$$= \frac{\cos\phi}{\sin\theta}\hat{e}_\theta + (3 - \cot\theta\sin\phi)\hat{e}_\phi$$

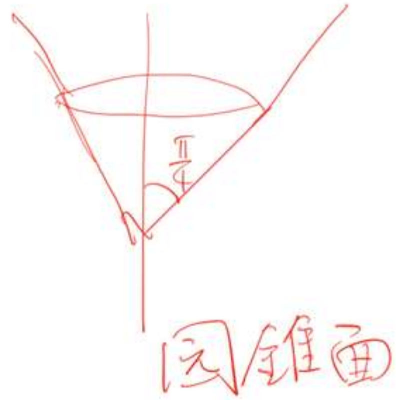
Problem 4

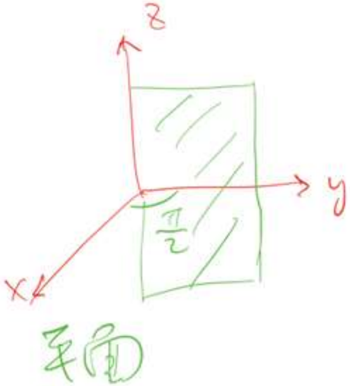
這個題目主要是要請同學們畫出方程式所指定的圖形：

在球坐標系當中請你畫出下面三個方程式所對應的曲面：

- $r = 3$
- $\theta = \pi/4$
- $\phi = pi/2$







在圓柱坐標系當中

- $\rho = 3$
- $\phi = \pi/3$
- $z = 3$