CHAPTER 3

粒子的波特性

Wave Properties of Particles

1924年Louise de Broglie假設及物質波

既然光可以具有「波粒二象性」，那麼實物粒子也可以具有「波粒二象性」！

物質波(matter wave)！

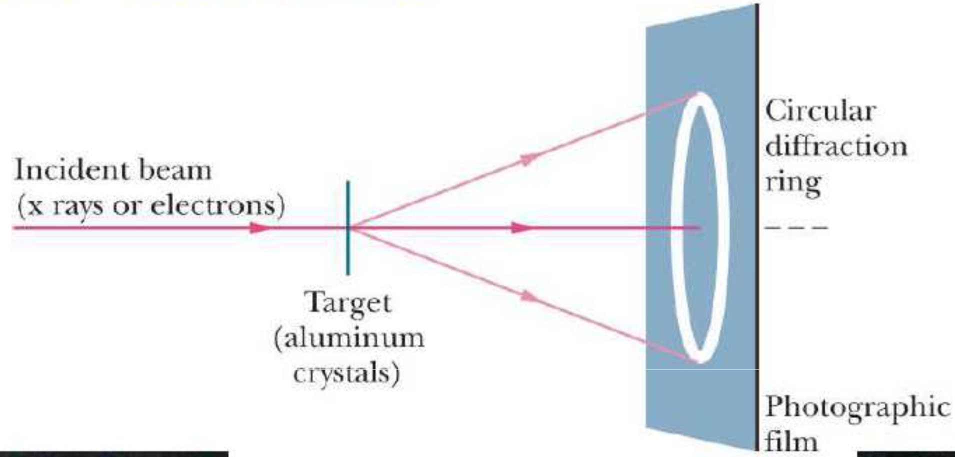
For photon $p = E / c = h\nu / c = h / \lambda$

For particle $p = h / \lambda$ **or** $\lambda = h / p$

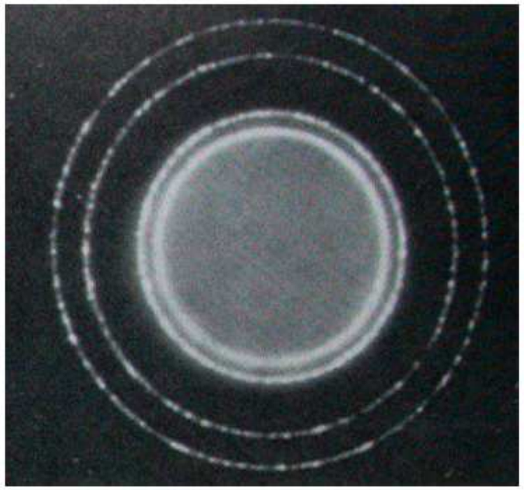
de Broglie (德布羅意)波長！

實驗證實

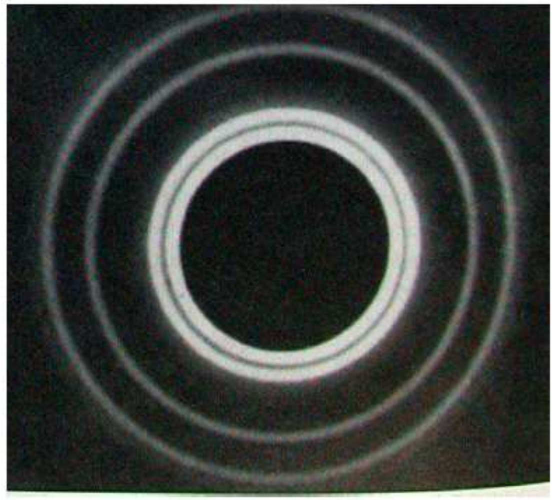
1927年 C. J. Davisson、L. Germer 和 G. P. Thomson 發現了電子的繞射現象



X-ray



Electron beam

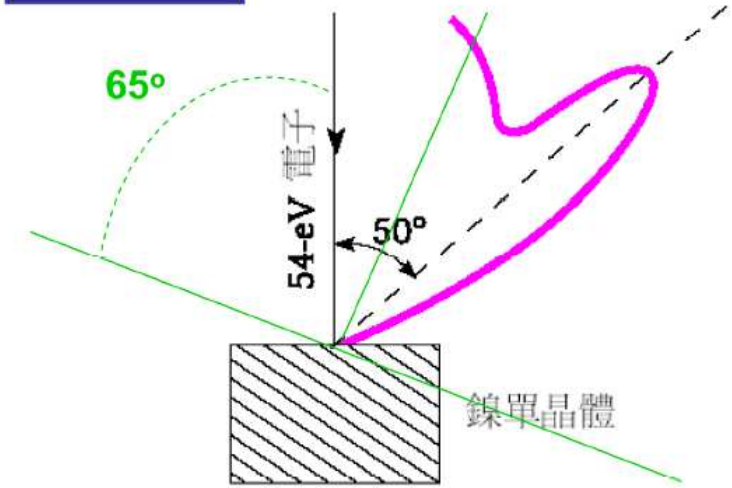


一道 54 eV 的電子垂直入射於鎳靶，而最大的電子分布位於和入射光成 50° 之處，入射光和散射光相對於此布拉格平面族的角度皆為 65°，如圖 3.8 所示，而藉由 x 射線繞射所量測到此平面族的平面間隔為 0.091 nm，繞射圖形中最大值的布拉格方程式為

$$n\lambda = 2d\sin\theta \tag{2.13}$$

此處 $d = 0.091\ \text{nm}$ 且 $\theta = 65°$。當 $n = 1$ 時，繞射電子的德布羅依波長為 去線

$$\lambda = 2d\sin\theta = (2)(0.091\ \text{nm})(\sin 65°) = \boxed{0.165\ \text{nm}}$$



現在我們利用德布羅依公式 $\lambda = h/\gamma m v$ 來計算電子的預期波長。54 eV 的電子動能和 0.51 MeV 的靜止能量相較之下非常地小，故我們假設 $\gamma = 1$，因為

$$\mathrm{KE} = \tfrac{1}{2} m v^2$$

電子動量 mv 為

$$\begin{aligned} mv &= \sqrt{2m\mathrm{KE}} \\ &= \sqrt{(2)(9.1\times10^{-31}\ \mathrm{kg})(54\ \mathrm{eV})(1.6\times10^{-19}\ \mathrm{J/eV})} \\ &= 4.0\times10^{-24}\ \mathrm{kg\cdot m/s} \end{aligned}$$

因此電子波長為

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{4.0\times10^{-24}\ \mathrm{kg\cdot m/s}} = 1.66\times10^{-10}\ \mathrm{m} = \boxed{0.166\ \mathrm{nm}}$$

這和所觀測到的波長 0.165 nm 相吻合，因此戴文生—基瑪實驗直接驗證了運動物體的德布羅依波特性。

現在新的實驗事實，迫使我們不得不承認微觀粒子除了具有粒子性以外，還具有波動性！亦即微觀粒子亦具有波粒二象性！

3.6 箱子中的粒子 Particle in a Box

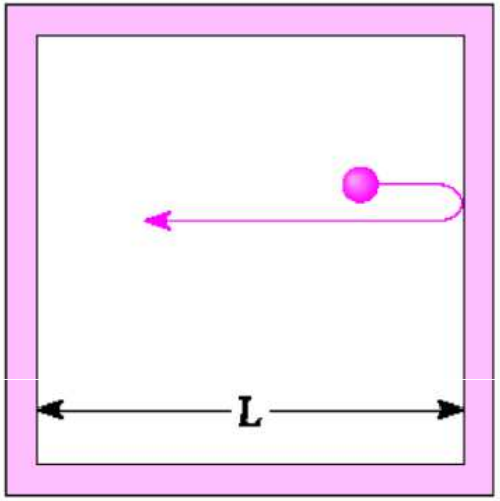


圖3.9 一粒子被限制於寬為 *L* 的箱中，假設粒子在箱壁之間以直線來回地運動。

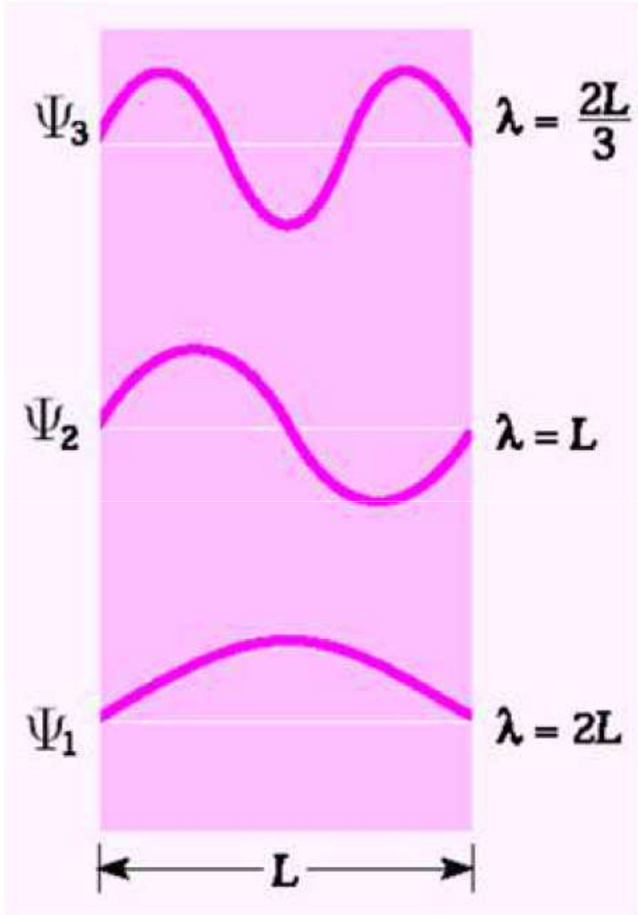
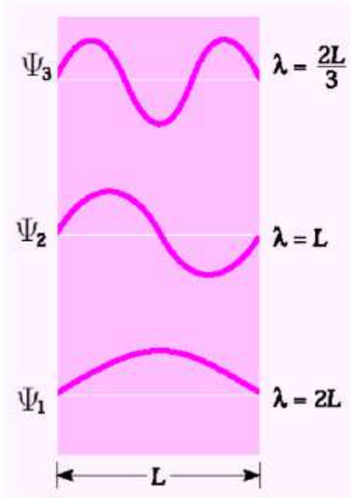


圖 3.10 被捕陷在寬 *L* 的箱中的粒子的波函數。

$$\boxed{L = n\frac{\lambda}{2}}$$



補陷粒子的德布羅依波長 $\boxed{\lambda = \frac{2L}{n}}$ $n = 1, 2, 3, \ldots$

因為 $mv = h/\lambda$，箱子寬度對於德布羅依波長 λ 的限制等效於對粒子動量的限制，故亦會限制其動能，動量 mv 的粒子動能為

$$\mathrm{KE} = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{2m\lambda^2}$$

允許波長為 $\lambda_n = 2L/n$，且因為在此模型中粒子並無位能，故其擁有的能量為

箱中的粒子 $$E_n = \frac{n^2h^2}{8mL^2} \qquad n = 1, 2, 3, \ldots \qquad (3.18)$$

每個允許的能量被稱為**能階**(energy level)，而決定能階 E_n 的整數 n 被稱為**量子數**(quantum number)。

$$E_n = \frac{n^2h^2}{8mL^2}$$

從式 (3.18) 中我們可以得到三個結論，這些結論可應用至任何被限制於特定空間中的粒子（甚至是在沒有清楚定義邊界的區域）

1.捕陷粒子並不能如同自由粒子，一樣擁有任意的能量，這個限制導致了波函數的限制，使得粒子僅能擁有特定的能量，而這些能量正和粒子質量以及如何被捕陷有關。

2.捕陷粒子能量不能為 0，因為粒子的德布羅依波長為 $\lambda = h/mv$，故速度為 0 意指波長為無限大。沒有任何方法能說明捕陷粒子的波長無限大，所以粒子至少必須擁有一些動能。

3.因為普朗克常數很小——只有 6.63×10^{-34} J·s ——能量的量子化僅當 m 和 L 非常小時才會顯著，這也就是為何在我們自己的經驗中對能量量子化並不清楚的原因。

如果讓電子一個個通過狹縫呢？

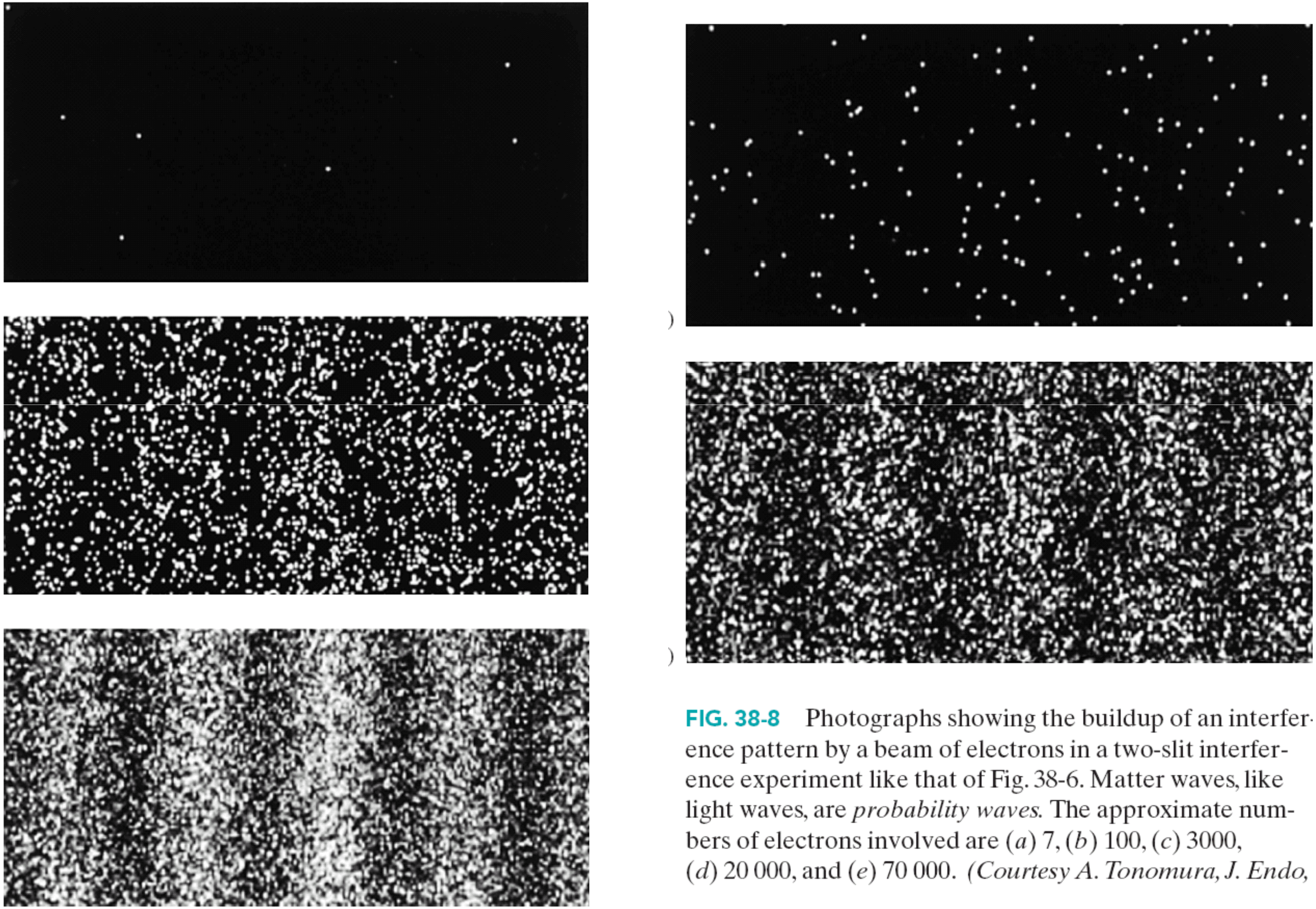
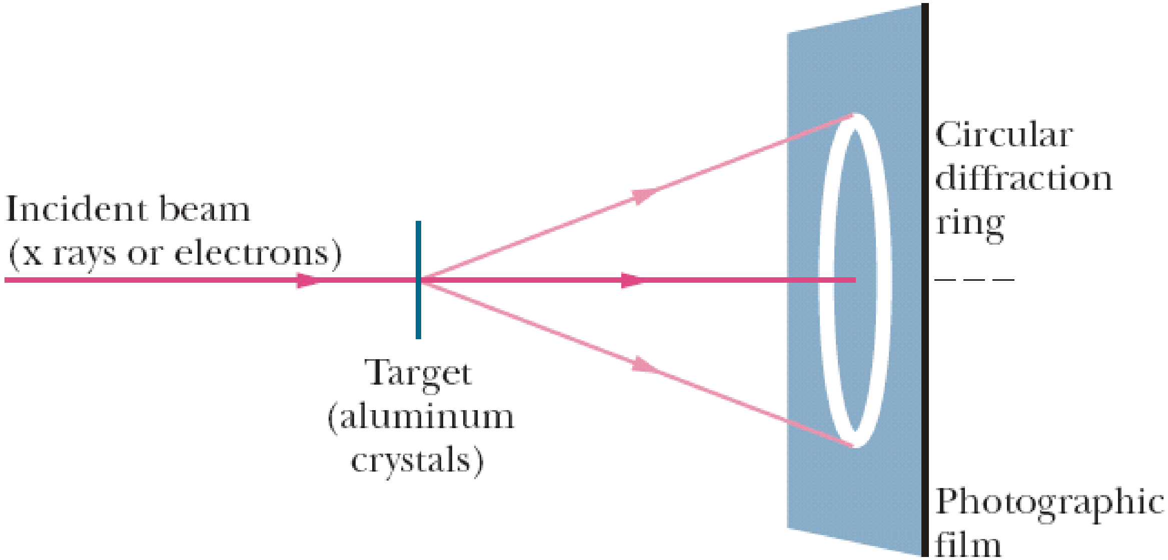
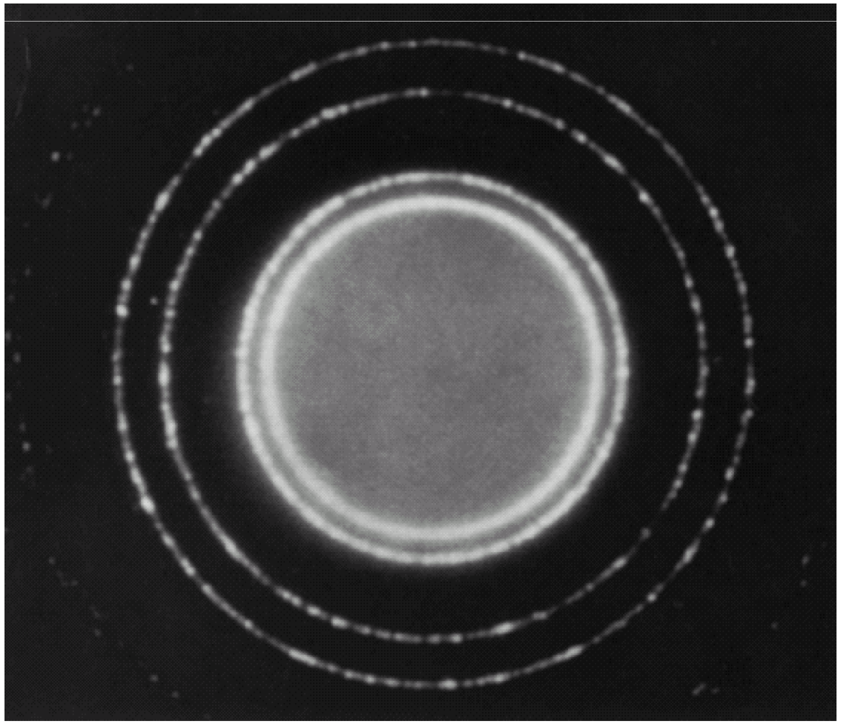
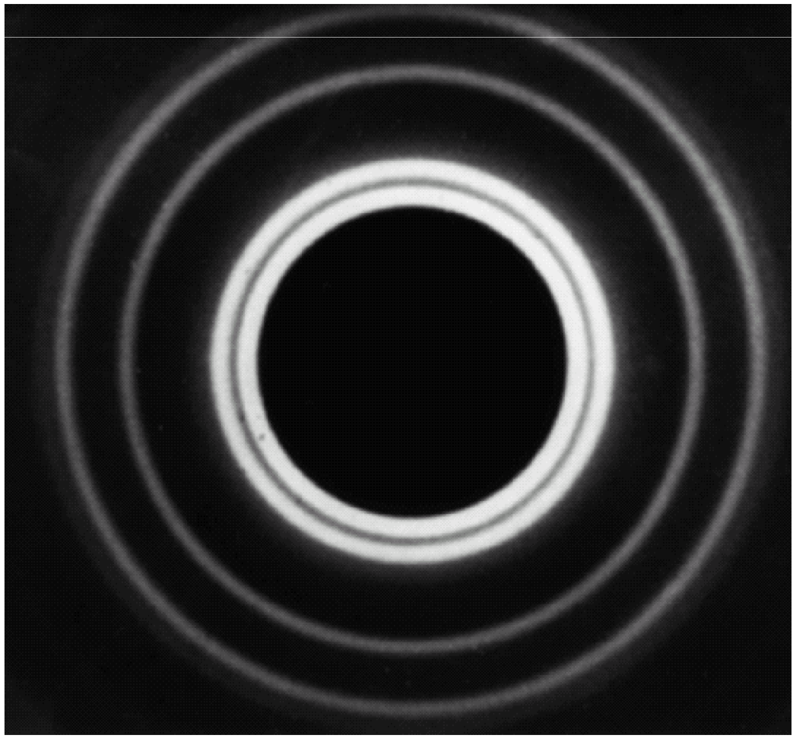


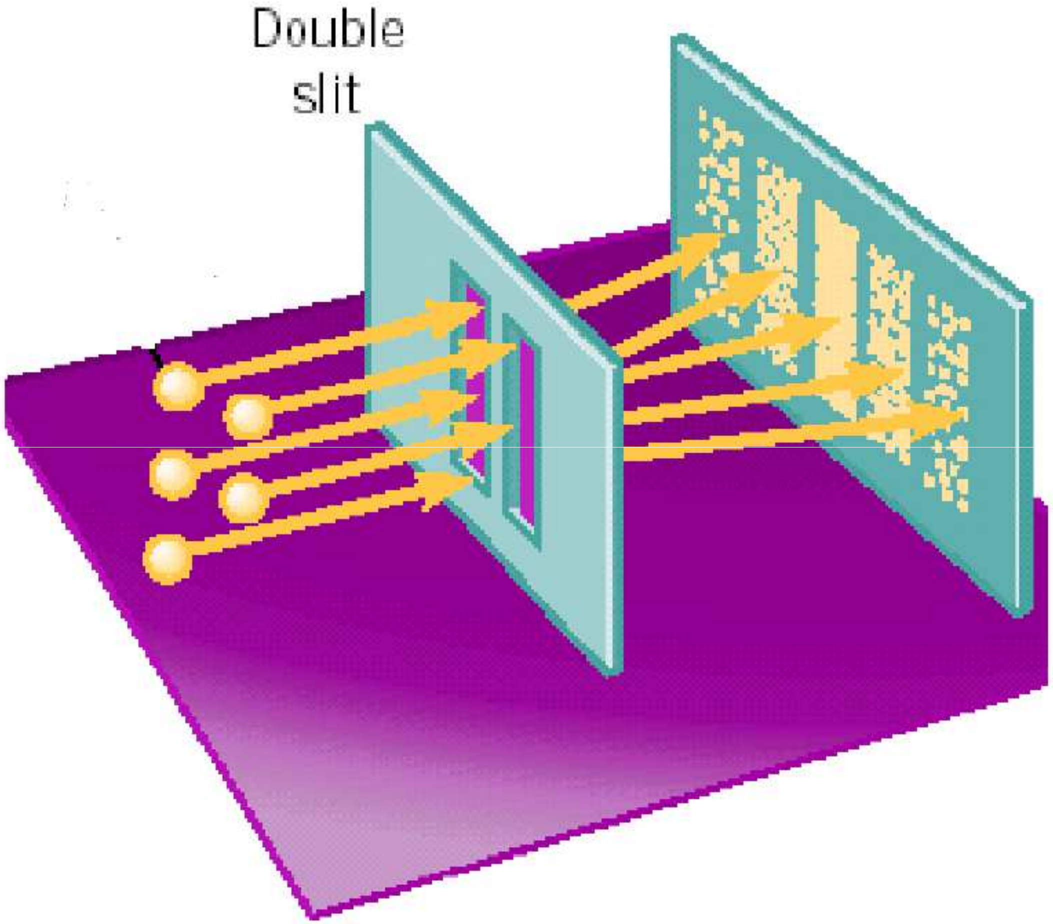
FIG. 38-8 Photographs showing the buildup of an interference pattern by a beam of electrons in a two-slit interference experiment like that of Fig. 38-6. Matter waves, like light waves, are *probability waves.* The approximate numbers of electrons involved are (*a*) 7, (*b*) 100, (*c*) 3000, (*d*) 20 000, and (*e*) 70 000. *(Courtesy A. Tonomura, J. Endo,*


Incident beam
(x rays or electrons)
Target
(aluminum
crystals)
Circular
diffraction
ring
Photographic
film





Probability wave（機率波）of electrons!



在底片上 r 點干涉條紋的強度 I (正比於波幅 Ψ 的平方)

正比於到達 r 點的電子數目 N，

因此亦正比於電子出現在 r 點的機率 P！

亦即，$P \propto |\Psi|^2$

物質波的波函數

物質波可用波函數 $\Psi(x,y,z,t)$ 來描述

波函數的強度 $|\Psi(x,y,z,t)|^2$ 的物理意義

波函數與其共軛複數的乘積

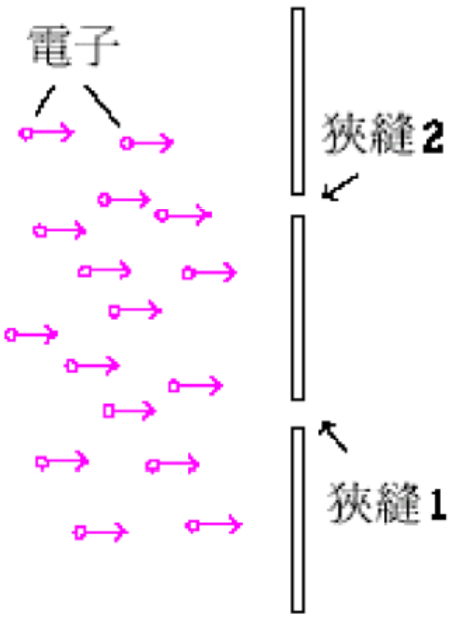
- t 時刻，出現在空間（x,y,z）點附近<u>**單位體積內**</u>的粒子數與總粒子數之比
- t 時刻，粒子出現在空間（x,y,z）點附近<u>**單位體積內**</u>的機率
- t 時刻，粒子在空間的機率<u>**密度**</u>分布

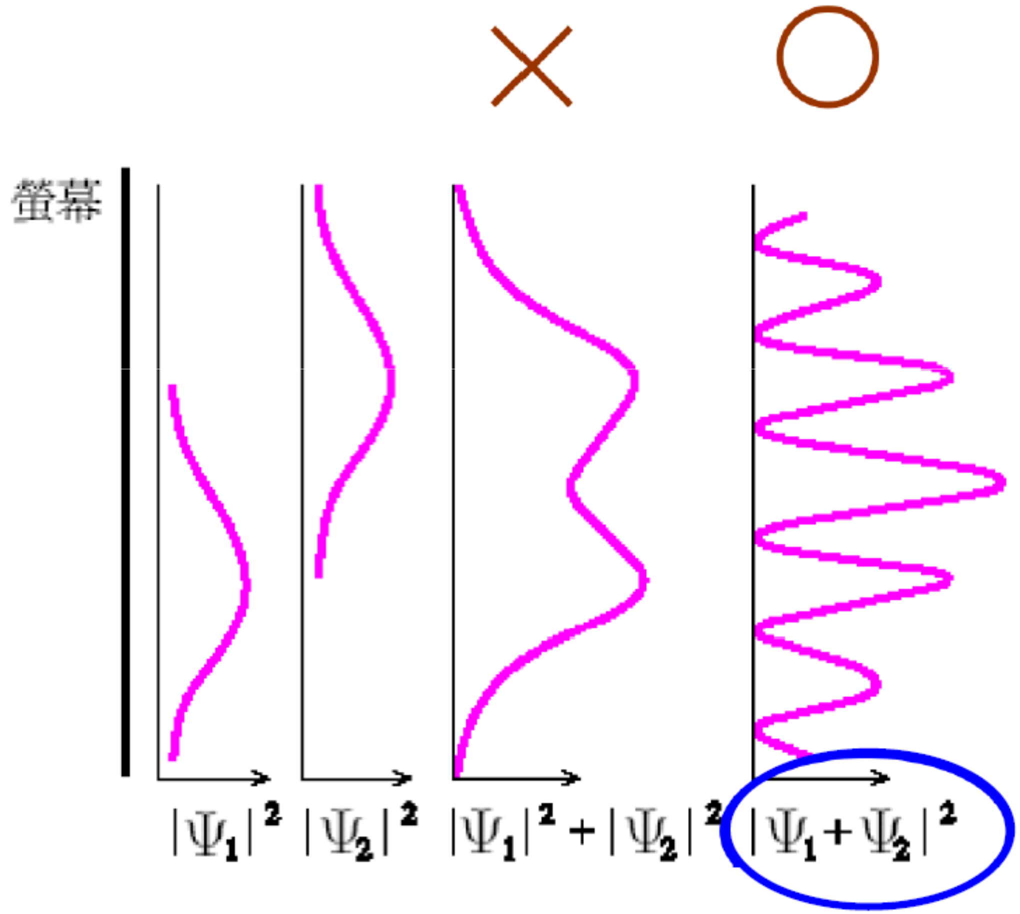
（1926 年Max Born）

對於雙狹縫實驗

$$\Psi = \Psi_1 + \Psi_2$$

$$|\Psi|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2|\Psi_1||\Psi_2|\cos\phi$$





3.3 描述一個波 Describing a Wave

德布羅依波速度 $$v_p = \nu\lambda = \left(\frac{\gamma mc^2}{h}\right)\left(\frac{h}{\gamma mv}\right) = \frac{c^2}{v} \tag{3.3}$$

因為粒子速度 v 必須比光速 c 小，德布羅依波總是比光速快！為瞭解這個不可預期的結果，我們必須瞭解**相速度**(phase velocity)和**群速度**(group velocity)的不同（相速度即為我們所稱之波速度）。

phase velocity : 單頻正弦波的速度
group velocity : 由許多單頻正弦波所合成的波整體移動的速度

波動公式 $$y = A\cos 2\pi\left(\nu t - \frac{x}{\lambda}\right) \tag{3.6}$$

然而，或許最常被用來描述波是式 (3.5) 的另一個形式，**角頻率** ω (angular frequency)和**波數** k(wave number)可以下列方程式定義

角頻率 $$\omega = 2\pi\nu \tag{3.7}$$

波數 $$k = \frac{2\pi}{\lambda} = \frac{\omega}{\upsilon_p} \tag{3.8}$$

以 ω 和 k 來表示式 (3.5) 如下

$$\boxed{\frac{\omega}{k} = \nu\lambda = \mathrm{v}_p}$$

波動公式 $$y = A\cos(\omega t - kx) \tag{3.9}$$

在三維空間中，k 變成垂直於波前的向量 $\mathbf{k}$，而以半徑向量 $\mathbf{r}$ 來取代 x，$\mathbf{k}\cdot\mathbf{r}$ 的純量乘積被用來取代式 (3.9) 中的 kx。

3.4 相速度與群速度 Phase and Group Velocities

組合而成的一群波，其速度未必會等於個別波的速度

- 對應於運動物體的de Broglie波振幅可反映出物體在特定時間和地點出現的機率。

$$y = A\cos(\omega t - kx)$$

很清楚地，德布羅依波無法以類似式 (3.9) 的公式來表示，因為它僅描述了具有相同振幅 A 之一連串無限長的波，相反地我們預期運動物體的波動表示對應到一**波包**(wave packet)或**波群**(wave group)，如圖 3.3 所示，物體出現的可能性和此波的振幅有關。

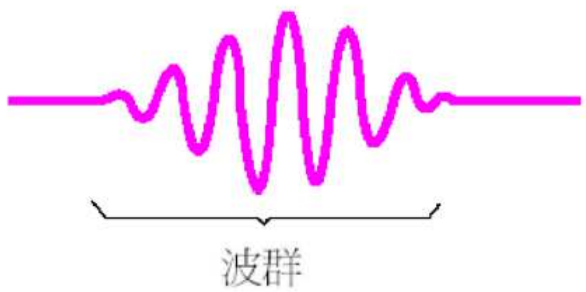
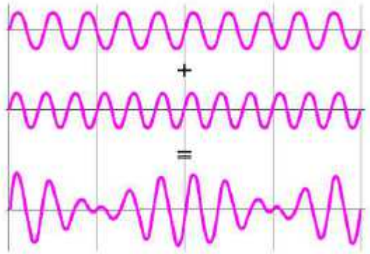


圖3.3 一個波群。

求得波群的速度 υ_g 並不困難，讓我們假設波群由兩個具有相同振幅 A、角頻率差為 $\Delta\omega$ 且波數差 Δk 的波組成，我們可以下列公式表示原來的波



$$\begin{cases} y_1 = A\cos(\omega t - kx) \\ y_2 = A\cos[(\omega + \Delta\omega)t - (k + \Delta k)x] \end{cases}$$

在任意時間 t 和地點 x 所產生 y 的位移為 y_1 和 y_2 的和，藉由恆等公式

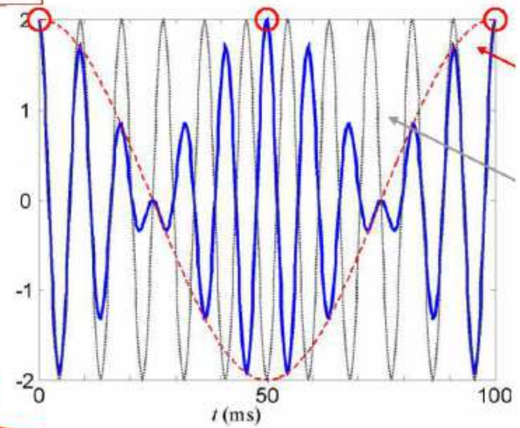
$$\begin{cases} \cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) \\ \cos(-\theta) = \cos\theta \end{cases}$$

和

我們發現

$$\begin{aligned} y &= y_1 + y_2 \\ &= 2A\cos\tfrac{1}{2}[(2\omega + \Delta\omega)t - (2k + \Delta k)x]\cos\tfrac{1}{2}(\Delta\omega t - \Delta k x) \end{aligned}$$

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= 2A\cos\tfrac{1}{2}[(2\omega+\Delta\omega)t-(2k+\Delta k)x]\cos\tfrac{1}{2}(\Delta\omega t-\Delta k x)
\end{aligned}
$$



因為 $\Delta\omega$, Δk 和 ω, k 相較之下很小，

$$
\begin{cases}
2\omega+\Delta\omega \approx 2\omega \\
2k+\Delta k \approx 2k
\end{cases}
$$

且

envelope

carrier

節拍
$$
y = 2A\cos(\omega t - k x)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \tag{3.10}
$$

式 (3.10) 代表角頻率 ω 和波數為 k 的波是藉由角頻率 $\frac{1}{2}\Delta\omega$ 和波數 $\frac{1}{2}\Delta k$ 的調變疊加而成。

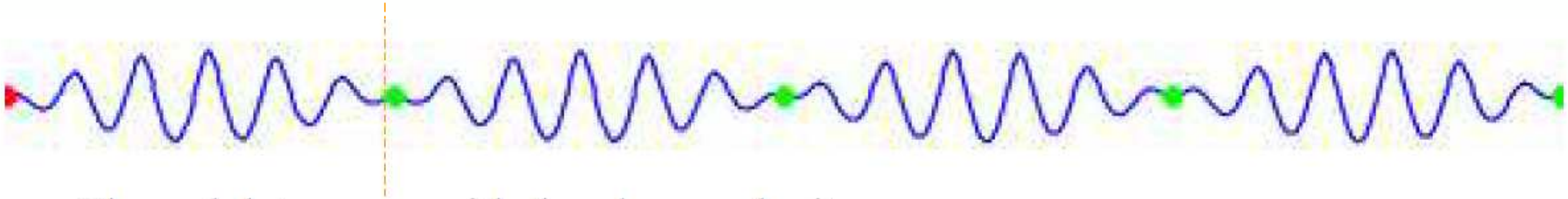
phase velocity (相速度) : $\upsilon_p = \dfrac{\omega}{k}$

group velocity (群速度) : $\upsilon_g = \dfrac{\Delta\omega}{\Delta k} \rightarrow \upsilon_g = \dfrac{d\omega}{dk}$

phase velocity (相速度) : $\upsilon_p = \dfrac{\omega}{k}$

group velocity (群速度) : $\upsilon_g = \dfrac{\Delta\omega}{\Delta k} \rightarrow \upsilon_g = \dfrac{d\omega}{dk}$

既然不同頻率的波跑得不一樣快，組合波的波形就為會變化，使波看起來有移動的感覺，此一移動之整體速度不必等於任何分量之波速，因此有群速度 與相速度之分。



The red dot moves with the phase velocity,
and the green dots propagate with the group velocity. (from Wikipedia)

See also http://140.122.141.1/demolab/phpBB/viewtopic.php?topic=14610

$$\boxed{\nu = \frac{\gamma mc^2}{h}} \qquad \boxed{\lambda = \frac{h}{p}}$$

對應於質量為 m 且以速度 v 行進的物體而言，其德布羅依波的角頻率和波數的關係為

$$\omega = 2\pi\nu = \frac{2\pi\gamma mc^2}{h}$$

德布羅依波的角頻率

$$= \frac{2\pi mc^2}{h\sqrt{1 - v^2/c^2}} \tag{3.14}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\gamma mv}{h}$$

德布羅依波的波數

$$= \frac{2\pi mv}{h\sqrt{1 - v^2/c^2}} \tag{3.15}$$

ω 和 k 皆為物體速度 v 的函數。

物體運動所產生的德布羅依波的群速度 υ_g 為

$$\upsilon_g = \frac{d\omega}{dk} = \frac{d\omega/d\upsilon}{dk/d\upsilon}$$

$$\boxed{\omega = \frac{2\pi mc^2}{h\sqrt{1-\upsilon^2/c^2}}} \qquad \boxed{k = \frac{2\pi m\upsilon}{h\sqrt{1-\upsilon^2/c^2}}}$$

$$\left\{\begin{aligned} \frac{d\omega}{d\upsilon} &= \frac{\cancel{2\pi m}\upsilon}{\cancel{h(1-\upsilon^2/c^2)^{3/2}}} \\ \frac{dk}{d\upsilon} &= \frac{\cancel{2\pi m}}{\cancel{h(1-\upsilon^2/c^2)^{3/2}}} \end{aligned}\right.$$

故群速度會變成

德布羅依波群速度 $\qquad \upsilon_g = \upsilon \qquad (3.16)$

運動物體之德布羅依波群速度和物體速度相同。

德布羅依波相速度 $\qquad \upsilon_p = \dfrac{\omega}{k} = \dfrac{c^2}{\upsilon} \qquad (3.3)$

這將超過物體的速度 υ 和光速 c，因為 $\upsilon < c$；然而 υ_p 並沒有物理意義，因為對應到物體的運動是波群運動，而非組成波群的波運動，且 $\upsilon_g < c$，所以對於德布羅依波而言，$\upsilon_p > c$ 並不會違背特殊相對論。

3.7 測不準原理 Uncertainty Principle

將運動物體當做波群暗示著我們所能量測的粒子特性，如位置和動量，有一基本的準確極限。

為瞭解其中的原因，讓我們來看圖 3.3 中的波群。對應到此波群的粒子在特定時間下可能位於波群中的任何地方。當然，機率密度 $|\Psi|^2$ 在群中間有最大值，所以最有可能在那裡出現，但是我們仍能在 $|\Psi|^2$ 不為 0 的任意地方發現粒子。

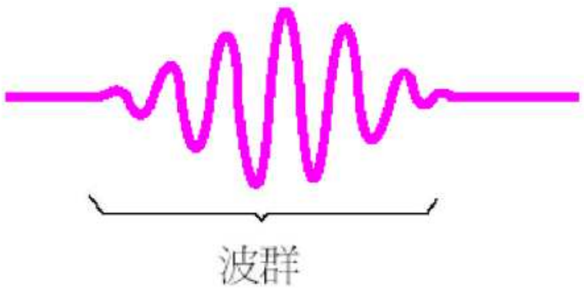


圖 3.3　一個波群。

1927年海森堡測不準原理
Heisenberg's uncertainty principle

某些成對的物理量(例如：位置與動量、時間與能量)不可能同時精準測得。

WERNER HEISENBERG
(1901 - 1976)

二者不準度的乘積不可能小於ħ/2！

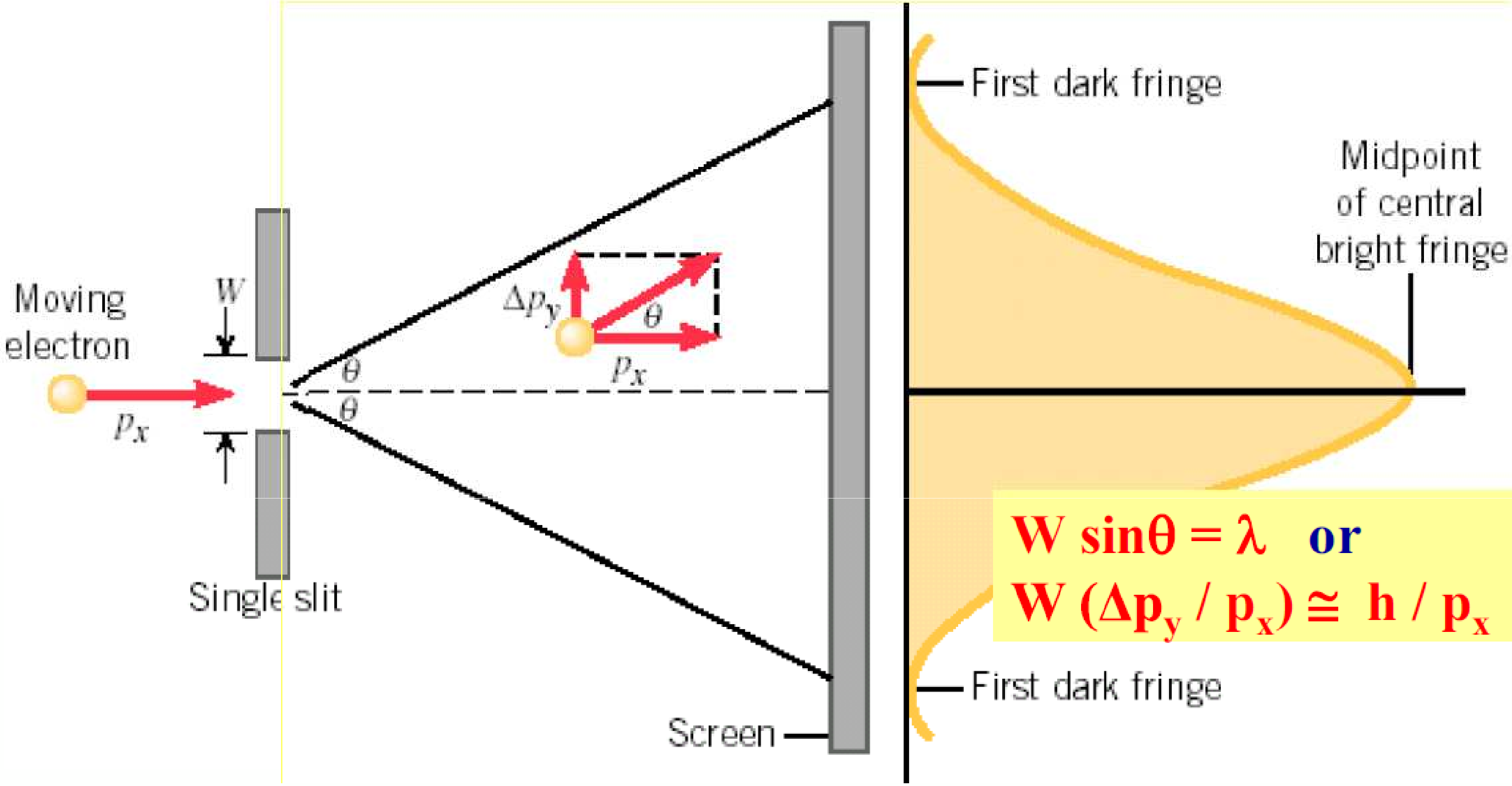
$$\Delta x \cdot \Delta p_x \geq \hbar / 2$$

$$\Delta y \cdot \Delta p_y \geq \hbar / 2$$

$$\Delta z \cdot \Delta p_z \geq \hbar / 2$$

h/2π這個量常在近代物理中出現，因為它剛好是角動量的基本單位。因此為了方便起見將 h/2π 簡寫成 **ħ (h-bar)**

測不準原理 — 1^{st} round



$W \sin\theta = \lambda$ or

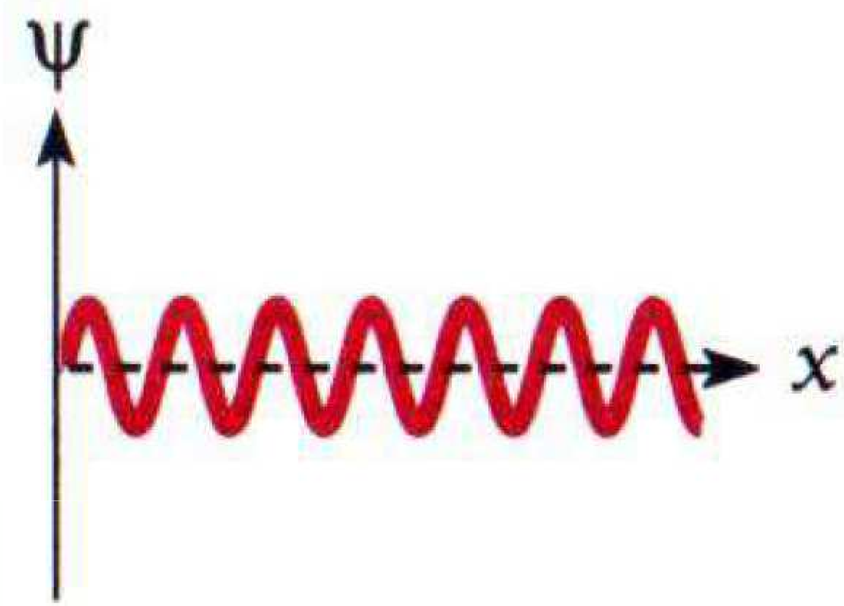
$W\,(\Delta p_y / p_x) \cong h / p_x$

$\Delta y = W$

$\Delta p_y \cong h / W$

$\Rightarrow \Delta y\, \Delta p_y \cong h$

測不準原理 — 2nd round



$$\psi(x) = A \sin(kx)$$

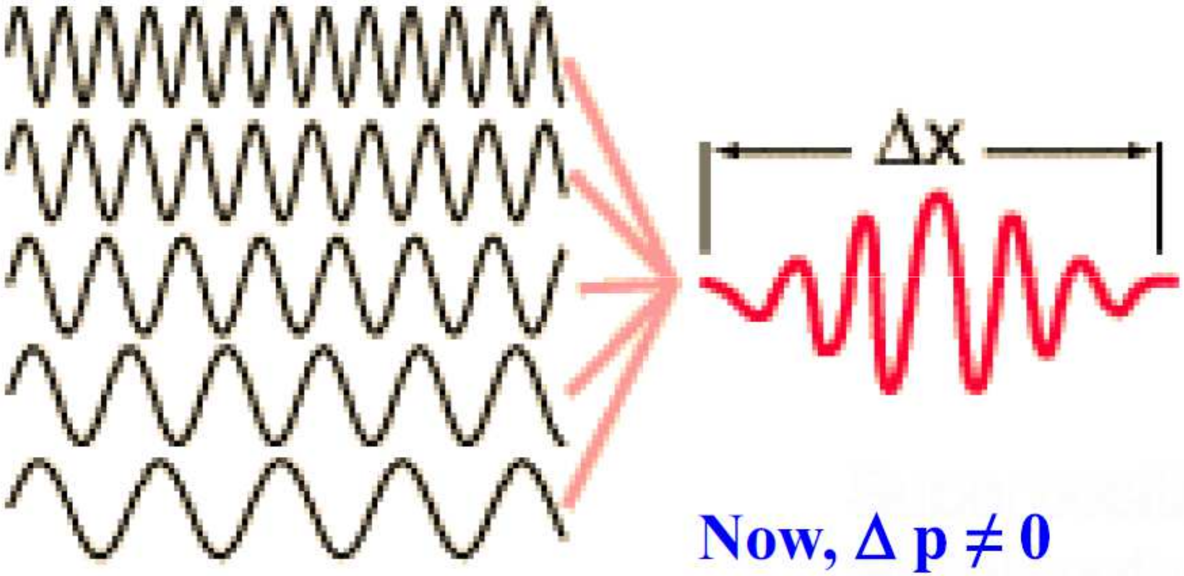
$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$$

That is, $\Delta p = 0$

⇒ a matter wave is spread out in space, and the position of the particle is poor define, i.e., $\Delta x = \infty$

To reduce Δx

$\Rightarrow$ superpose many wavelengths to form a reasonably well-localized wave packet （波包）



Now, $\Delta p \neq 0$

3.9 測不準原理應用
例 3.8

- 一個氫原子的半徑為 5.3×10^{-11} m 。運用測不準原理估算在此原子中電子所需具備的極小能量。

解

這裡我們發現 $\Delta x = 5.3\times10^{-11}$ m 。

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq 9.9\times10^{-25}\ \mathrm{kg\cdot m/s}$$

具有這種大小等級動量的電子表現和古典的粒子相似，它的動能為

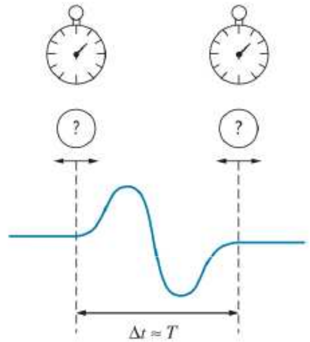
$$\mathrm{KE} = \frac{p^2}{2m} \geq \frac{(9.9\times10^{-25}\ \mathrm{kg\cdot m/s})^2}{(2)(9.1\times10^{-31}\ \mathrm{kg})} \geq 5.4\times10^{-19}\ \mathrm{J}$$

也就是 3.4 eV。在氫原子最低能階的電子動能實際上是 13.6 eV。

能量與時間

另一種型式的測不準原理與能量和時間有關。我們也許希望量測原子在時間間隔 Δt 中所發射出的能量 E。如果能量以電磁波的形式存在，則有限時間使得準確度受到限制而無法準確地量測波的頻率 ν。讓我們假設量測波群的波數目時最小的測不準量是一個波。因為所研究波的頻率等於我們所數到的數目除以時間間隔，我們對頻率的測不準量 $\Delta\nu$ 為

$$\Delta\nu = \nu' - \nu = \frac{n'}{\Delta t} - \frac{n}{\Delta t} = \frac{\Delta n}{\Delta t} \geq \frac{1}{\Delta t}$$



對應的能量測不準量為

$$\Delta E = h\Delta\nu$$

所以

$$\Delta E \geq \frac{h}{\Delta t} \qquad 或 \qquad \Delta E\Delta t \geq h$$

由波群的特性可得到一個更精確的計算，所得結果變為

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}} \quad (3.22)$$

能量與時間的測不準量 $$\Delta E\Delta t \geq \frac{\hbar}{2}$$ Remember ! (3.26)

CHAPTER 3

Wave Properties of Particles

3.1 DE BROGLIE WAVES

A moving body behaves in certain ways as though it has a wave nature

A photon of light of frequency ν has the momentum

$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$

since $\lambda\nu = c$. The wavelength of a photon is therefore specified by its momentum according to the relation

Photon wavelength

$$\lambda = \frac{h}{p} \tag{3.1}$$

De Broglie suggested that Eq. (3.1) is a completely general one that applies to material particles as well as to photons. The momentum of a particle of mass m and velocity $\boldsymbol{v}$ is $p = \gamma m v$, and its **de Broglie wavelength** is accordingly

De Broglie wavelength

$$\lambda = \frac{h}{\gamma m v} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.2}$$

Find the de Broglie wavelength of a 1.00-MeV proton. Is a relativistic calculation needed?

The proton's kinetic energy is only about 0.1% of its rest energy, so a nonrelativistic calculation will suffice. The wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2\,m\,\mathrm{KE}}} = \frac{hc}{\sqrt{2\,(mc^2)\,\mathrm{KE}}}$$

$$= \frac{1.240 \times 10^{-12}\ \mathrm{MeV\cdot m}}{\sqrt{2\,(939.3\ \mathrm{MeV})\,(1.00\ \mathrm{MeV})}} = 2.86 \times 10^{-14}\ \mathrm{m}.$$

Note the conversion of units in the product hc in the above calculation.

Example 3.1

Find the de Broglie wavelengths of (*a*) a 46-g golf ball with a velocity of 30 m/s, and (*b*) an electron with a velocity of 10^7 m/s.

Solution

(*a*) Since $\boldsymbol{v} \ll c$, we can let $\gamma = 1$. Hence

$$\lambda = \frac{h}{m\boldsymbol{v}} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(0.046\ \text{kg})(30\ \text{m/s})} = 4.8 \times 10^{-34}\ \text{m}$$

The wavelength of the golf ball is so small compared with its dimensions that we would not expect to find any wave aspects in its behavior.

(*b*) Again $\boldsymbol{v} \ll c$, so with $m = 9.1 \times 10^{-31}$ kg, we have

$$\lambda = \frac{h}{m\boldsymbol{v}} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.1 \times 10^{-31}\ \text{kg})(10^7\ \text{m/s})} = 7.3 \times 10^{-11}\ \text{m}$$

The dimensions of atoms are comparable with this figure—the radius of the hydrogen atom, for instance, is 5.3×10^{-11} m. It is therefore not surprising that the wave character of moving electrons is the key to understanding atomic structure and behavior.

3.2 WAVES OF WHAT?

Waves of probability

Max Born (1882–1970)

The probability of experimentally finding the body described by the wave function Ψ at the point x, y, z, at the time t is proportional to the value of $|\Psi|^2$ there at t.

This interpretation was first made by Max Born in 1926.

A large value of $|\Psi|^2$ means the strong possibility of the body's presence, while a small value of $|\Psi|^2$ means the slight possibility of its presence. As long as $|\Psi|^2$ is not actually 0 somewhere, however, there is a definite chance, however small, of detecting it there. This interpretation was first made by Max Born in 1926.

3.3 DESCRIBING A WAVE

A general formula for waves

If we call the de Broglie wave velocity v_p, we can apply the usual formula

$$v_p = \nu\lambda$$

to find v_p. The wavelength λ is simply the de Broglie wavelength $\lambda = h/\gamma m v$. To find the frequency, we equate the quantum expression $E = h\nu$ with the relativistic formula for total energy $E = \gamma mc^2$ to obtain

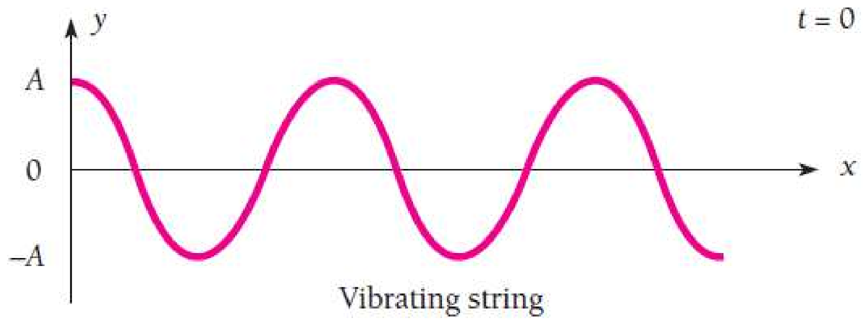
$$h\nu = \gamma mc^2$$

$$\nu = \frac{\gamma mc^2}{h}$$

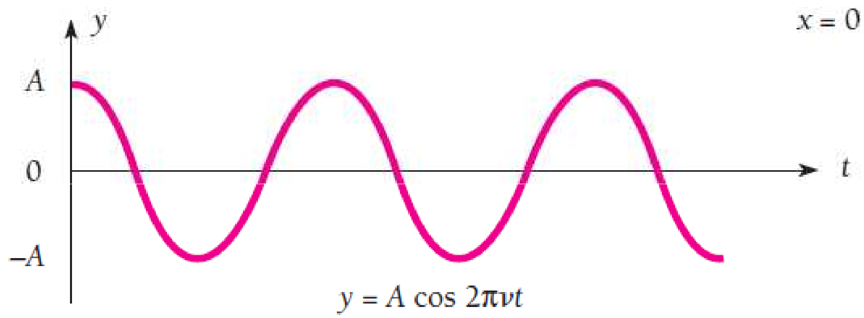
The de Broglie wave velocity is therefore

De Broglie phase velocity

$$v_p = \nu\lambda = \left(\frac{\gamma mc^2}{h}\right)\left(\frac{h}{\gamma m v}\right) = \frac{c^2}{v} \tag{3.3}$$



(a)



Wave formula

$$y = A \cos 2\pi\nu\left(t - \frac{x}{v_p}\right) \tag{3.5}$$

As a check, we note that Eq. (3.5) reduces to Eq. (3.4) at $x = 0$.

Equation (3.5) may be rewritten

$$y = A \cos 2\pi\left(\nu t - \frac{\nu x}{v_p}\right)$$

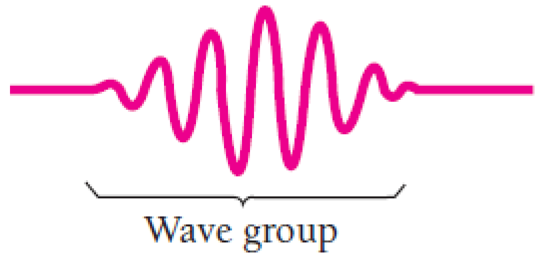
$$y = A \cos 2\pi\left(\nu t - \frac{x}{\lambda}\right) \qquad y = A \cos(\omega t - kx)$$

Angular frequency $\omega = 2\pi\nu$

Wave number $k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p}$

3.4 PHASE AND GROUP VELOCITIES

A group of waves need not have the same velocity as the waves themselves



we expect the wave representation of a moving body to correspond to a **wave packet,** or **wave group, like that shown in Fig. 3.3, whose waves have amplitudes upon which** the likelihood of detecting the body depends.

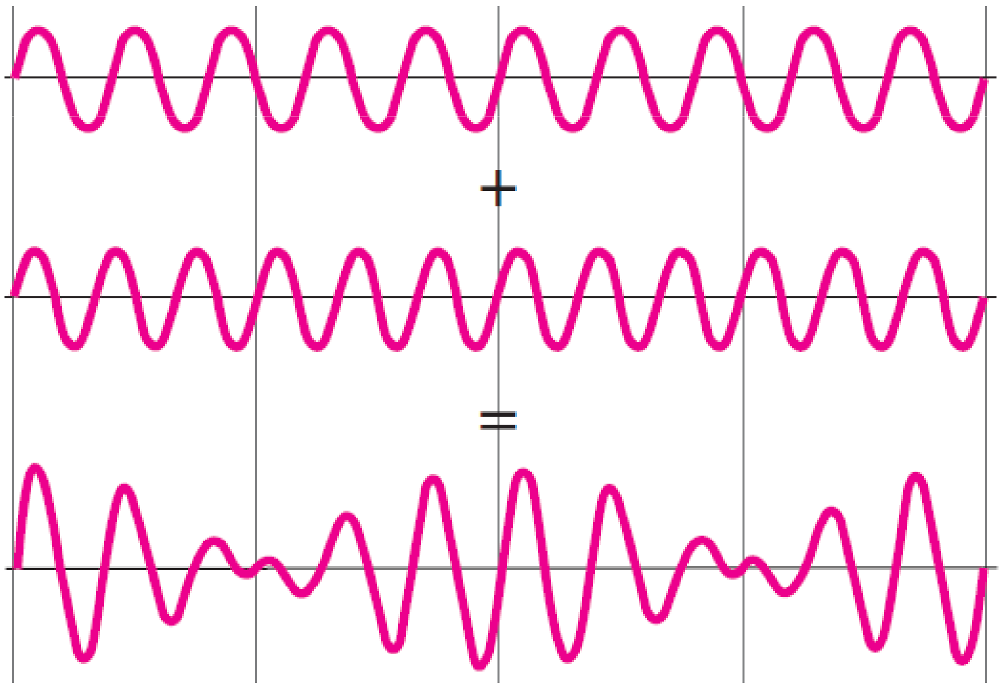
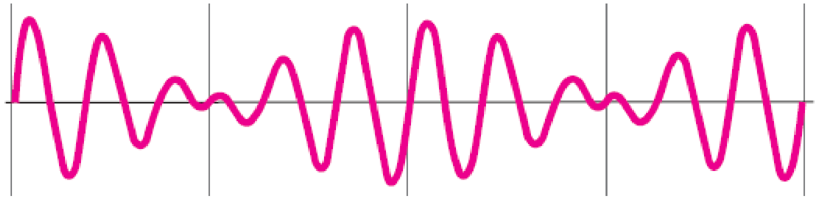


Figure 3.4 Beats are produced by the superposition of two waves with different frequencies.

$$y_1 = A \cos(\omega t - kx)$$

$$y_2 = A \cos[(\omega + \Delta\omega)t - (k + \Delta k)x]$$



$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta) \qquad \cos(-\theta) = \cos\theta$$

$$\begin{aligned} y &= y_1 + y_2 \\ &= 2A\cos\tfrac{1}{2}[(2\omega + \Delta\omega)t - (2k + \Delta k)x]\cos\tfrac{1}{2}(\Delta\omega\, t - \Delta k\, x) \end{aligned}$$

Since $\Delta\omega$ and Δk are small compared with ω and k respectively,

$$2\omega + \Delta\omega \approx 2\omega$$

$$2k + \Delta k \approx 2k$$

and so

Beats $$y = 2A\cos(\omega t - kx)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$$

Equation (3.10) represents a wave of angular frequency ω and wave number k that has superimposed upon it a modulation of angular frequency $\frac{1}{2}\Delta\omega$ and of wave number $\frac{1}{2}\Delta k$. The effect of the modulation is to produce successive wave groups.

The phase velocity v_p is

$$h\nu = \gamma mc^2$$

Phase velocity $$v_p = \frac{\omega}{k}$$ $$\nu = \frac{\gamma mc^2}{h} \quad (3.11)$$

and the velocity v_g of the wave groups is

$$v_p = \nu\lambda = \left(\frac{\gamma mc^2}{h}\right)\left(\frac{h}{\gamma mv}\right) = \frac{c^2}{v}$$

Group velocity $$v_g = \frac{\Delta\omega}{\Delta k} \quad (3.12)$$

When ω and k have continuous spreads instead of the two values in the preceding discussion, the group velocity is instead given by

Group velocity $$v_g = \frac{d\omega}{dk} \quad (3.13)$$

$$\omega = 2\pi\nu = \frac{2\pi\gamma mc^2}{h} = \frac{2\pi mc^2}{h\sqrt{1 - v^2/c^2}}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\gamma mv}{h} = \frac{2\pi mv}{h\sqrt{1 - v^2/c^2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\frac{d\omega}{dv} = \frac{2\pi mv}{h(1 - v^2/c^2)^{3/2}}$$

$$\frac{dk}{dv} = \frac{2\pi m}{h(1 - v^2/c^2)^{3/2}}$$

the group velocity turns out to be

$$v_g = v$$

$$v_p = \frac{\omega}{k} = \frac{c^2}{v}$$

Example 3.3

An electron has a de Broglie wavelength of 2.00 pm $= 2.00 \times 10^{-12}$ m. Find its kinetic energy and the phase and group velocities of its de Broglie waves.

Solution

(*a*) The first step is to calculate pc for the electron, which is

$$pc = \frac{hc}{\lambda} = \frac{(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{2.00 \times 10^{-12}\ \text{m}} = 6.20 \times 10^{5}\ \text{eV}$$

$$= 620\ \text{keV}$$

The rest energy of the electron is $E_0 = 511$ keV, so

$$\text{KE} = E - E_0 = \sqrt{E_0^2 + (pc)^2} - E_0 = \sqrt{(511\ \text{keV})^2 + (620\ \text{keV})^2} - 511\ \text{keV}$$

$$= 803\ \text{keV} - 511\ \text{keV} = 292\ \text{keV}$$

(*b*) The electron velocity can be found from

$$E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$$

$$v_p = \frac{c^2}{v} = \frac{c^2}{0.771c} = 1.30c$$

$$v_g = v = 0.771c$$

to be

$$v = c\sqrt{1 - \frac{E_0^2}{E^2}} = c\sqrt{1 - \left(\frac{511\ \text{keV}}{803\ \text{keV}}\right)^2} = 0.771c$$

3.5 PARTICLE DIFFRACTION

An experiment that confirms the existence of de Broglie waves

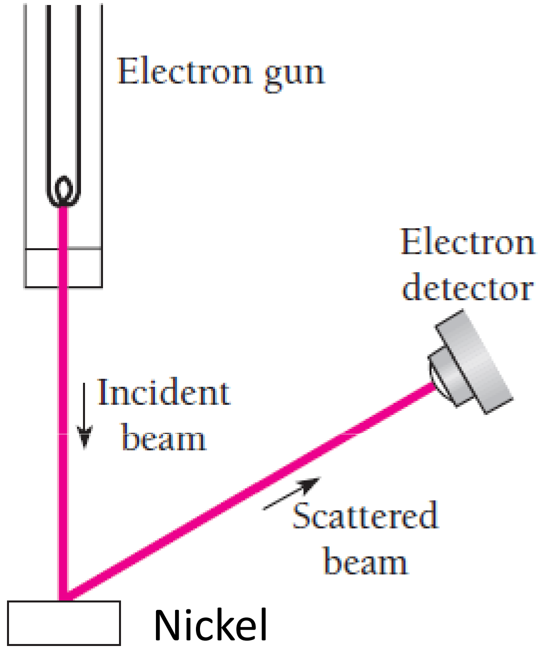
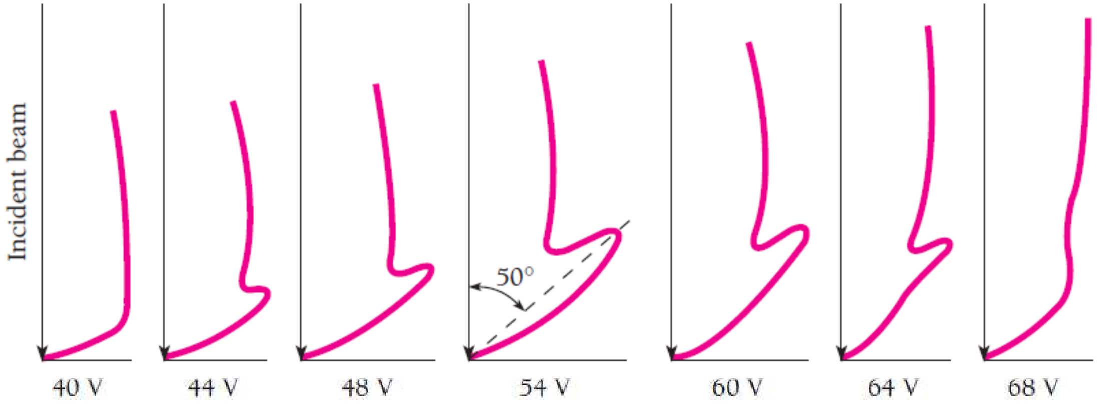


Figure 3.6 The Davisson-Germer experiment.

the intensity at any angle is proportional to the distance of the curve at that angle from the point of scattering.



Results of the Davisson-Germer experiment, showing how the number of scattered electrons varied with the angle between the incoming beam and the crystal surface. The Bragg planes of atoms in the crystal were not parallel to the crystal surface, so the angles of incidence and scattering relative to one family of these planes were both 65°.

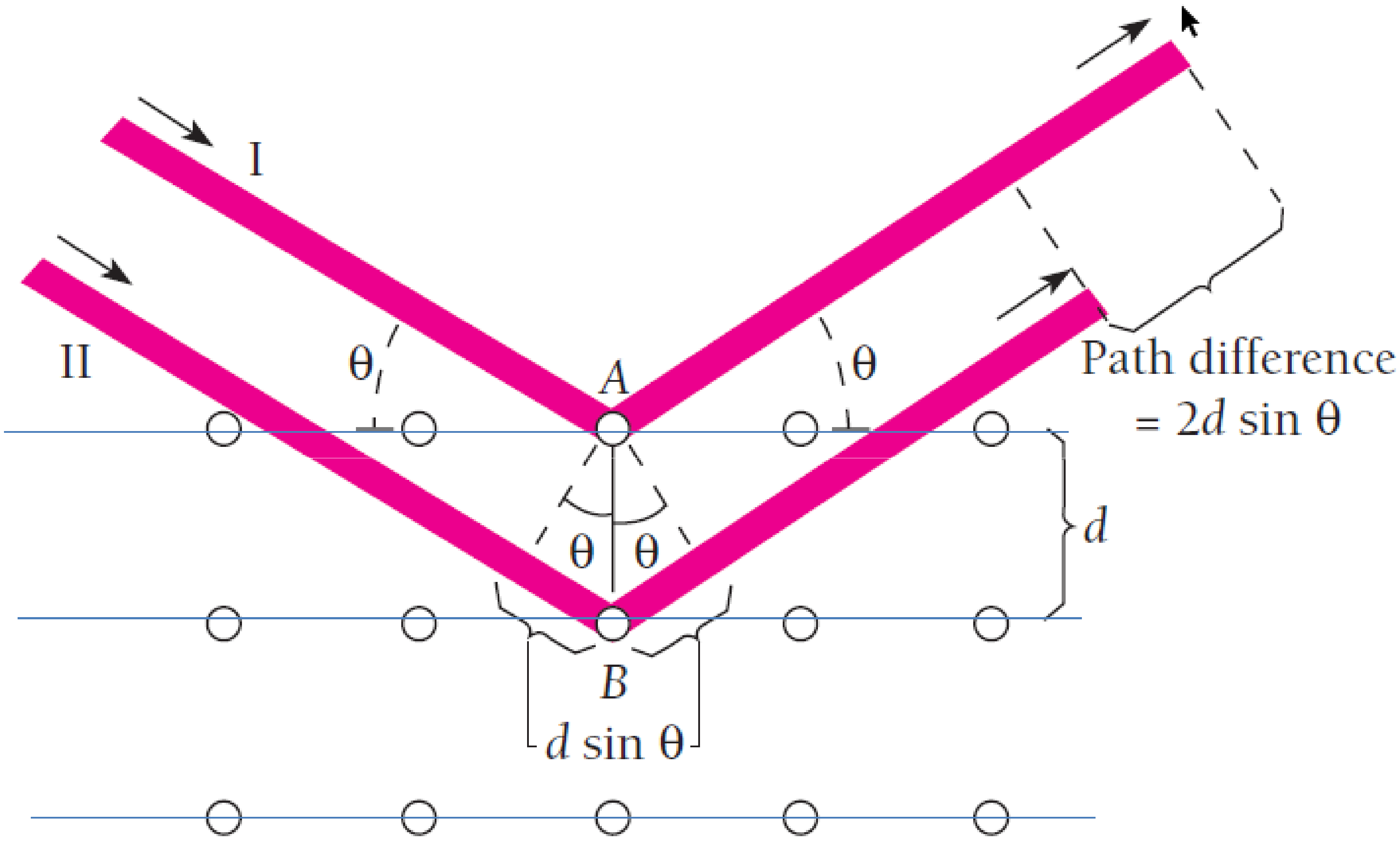
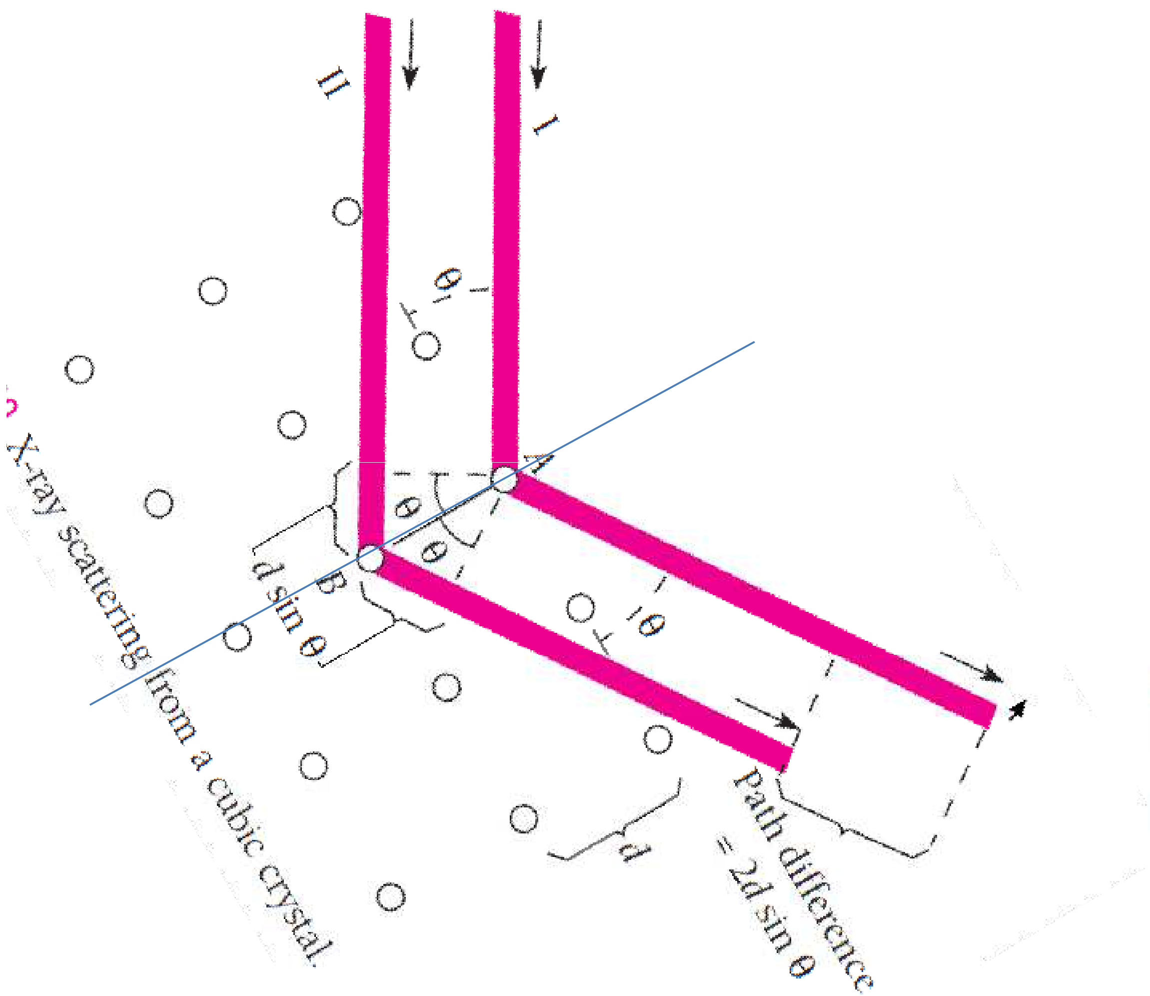


Figure 2.20 X-ray scattering from a cubic crystal.



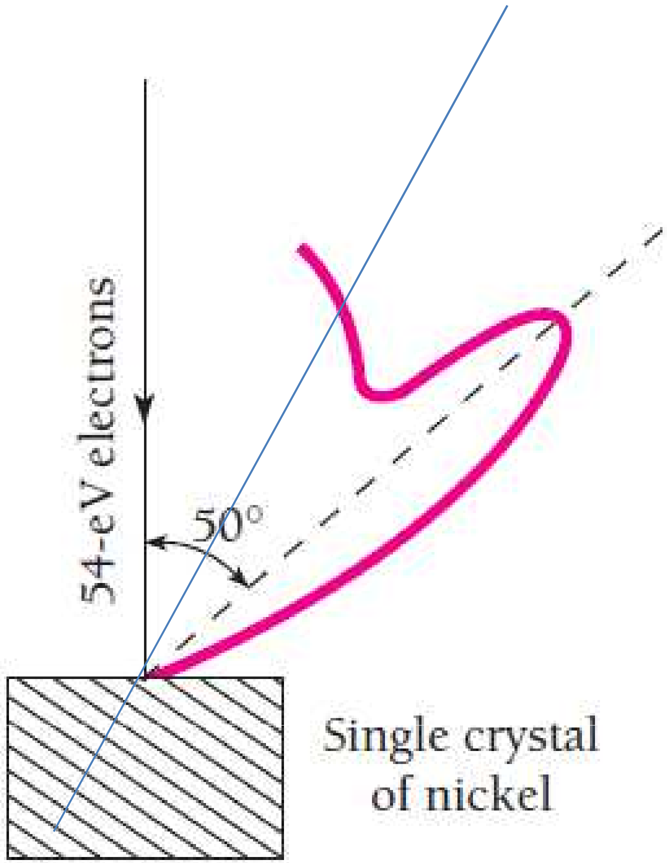


Figure 3.8 The diffraction of the de Broglie waves by the target is responsible for the results of Davisson and Germer.

spacing of the planes in this family, which can be measured by x-ray diffraction, is 0.091 nm. The Bragg equation for maxima in the diffraction pattern is

$$n\lambda = 2d \sin \theta \tag{2.13}$$

Here $d = 0.091$ nm and $\theta = 65°$. For $n = 1$ the de Broglie wavelength λ of the diffracted electrons is

$$\lambda = 2d \sin \theta = (2)(0.091 \text{ nm})(\sin 65°) = 0.165 \text{ nm}$$

Now we use de Broglie's formula $\lambda = h/\gamma mv$ to find the expected wavelength of the electrons. The electron kinetic energy of 54 eV is small compared with its rest energy mc^2 of 0.51 MeV, so we can let $\gamma = 1$. Since

$$\text{KE} = \tfrac{1}{2}mv^2$$

the electron momentum mv is

$$\begin{aligned} mv &= \sqrt{2m\text{KE}} \\ &= \sqrt{(2)(9.1 \times 10^{-31} \text{ kg})(54 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})} \\ &= 4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s} \end{aligned}$$

The electron wavelength is therefore

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s}} = 1.66 \times 10^{-10} \text{ m} = 0.166 \text{ nm}$$

which agrees well with the observed wavelength of 0.165 nm. The Davisson-Germer

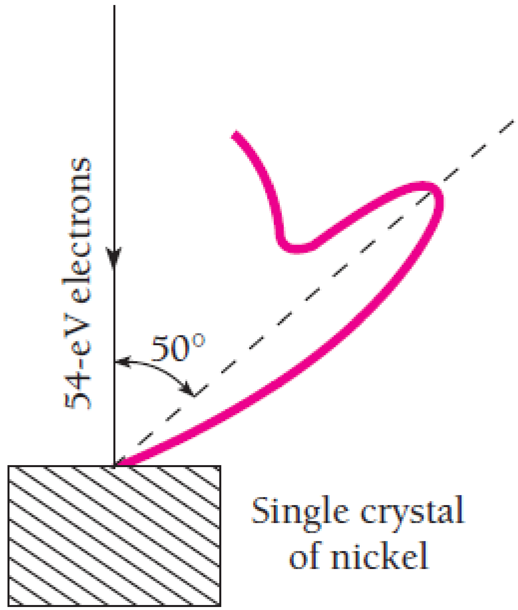


Figure 3.8 The diffraction of the de Broglie waves by the target is responsible for the results of Davisson and Germer.

3.6 PARTICLE IN A BOX

Why the energy of a trapped particle is quantized

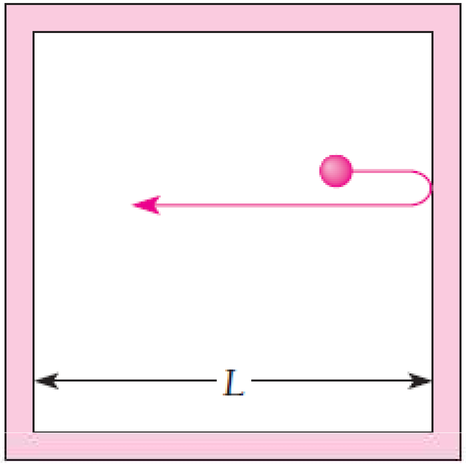
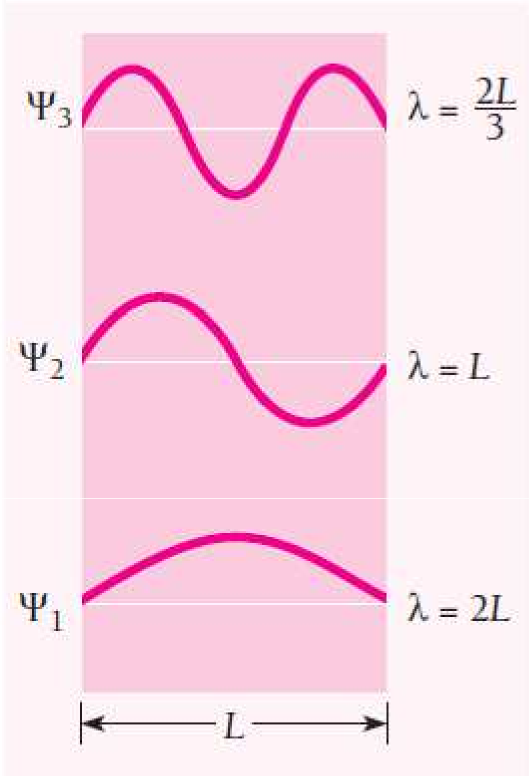


Figure 3.9 A particle confined to a box of width L. The particle is assumed to move back and forth along a straight line between the walls of the box.



Wave functions of a particle trapped in a box *L wide.*

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots$$

$$KE = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{2m\lambda^2}$$

$$E_n = \frac{n^2h^2}{8mL^2} \qquad n = 1, 2, 3, \ldots$$

We can draw three general conclusions:

1 A trapped particle cannot have an arbitrary energy, as a free particle can. The fact of its confinement leads to restrictions on its wave function that allow the particle to have only certain specific energies and no others. Exactly what these energies are depends on the mass of the particle and on the details of how it is trapped.

2 A trapped particle cannot have zero energy. Since the de Broglie wavelength of the particle is $\lambda = h/mv$, a speed of $v = 0$ means an infinite wavelength. But there is no way to reconcile an infinite wavelength with a trapped particle, so such a particle must have at least some kinetic energy. The exclusion of $E = 0$ for a trapped particle, like the limitation of E to a set of discrete values, is a result with no counterpart in classical physics, where all non-negative energies, including zero, are allowed.

3 Because Planck's constant is so small—only 6.63×10^{-34} J · s—quantization of energy is conspicuous only when m and L are also small. This is why we are not aware of energy quantization in our own experience. Two examples will make this clear.

Example 3.4

An electron is in a box 0.10 nm across, which is the order of magnitude of atomic dimensions. Find its permitted energies.

Solution

Here $m = 9.1 \times 10^{-31}$ kg and $L = 0.10$ nm $= 1.0 \times 10^{-10}$ m, so that the permitted electron energies are

$$E_n = \frac{(n^2)(6.63 \times 10^{-34}\ \mathrm{J \cdot s})^2}{(8)(9.1 \times 10^{-31}\ \mathrm{kg})(1.0 \times 10^{-10}\ \mathrm{m})^2} = 6.0 \times 10^{-18} n^2\ \mathrm{J}$$
$$= 38n^2\ \mathrm{eV}$$

The minimum energy the electron can have is 38 eV, corresponding to $n = 1$. The sequence of energy levels continues with $E_2 = 152$ eV, $E_3 = 342$ eV, $E_4 = 608$ eV, and so on (Fig. 3.11). If such a box existed, the quantization of a trapped electron's energy would be a prominent feature of the system. (And indeed energy quantization is prominent in the case of an atomic electron.)

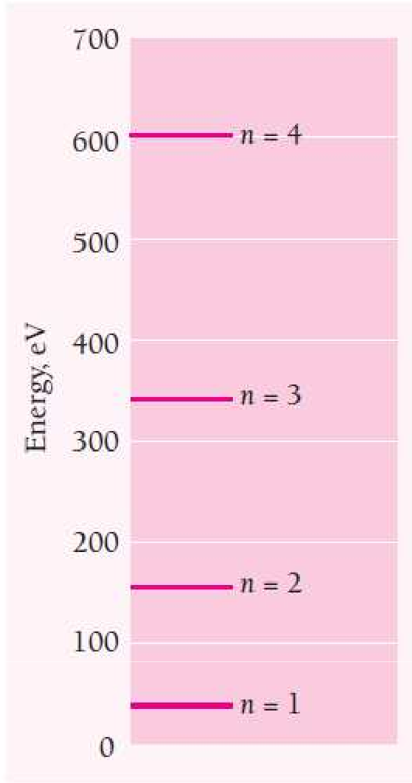
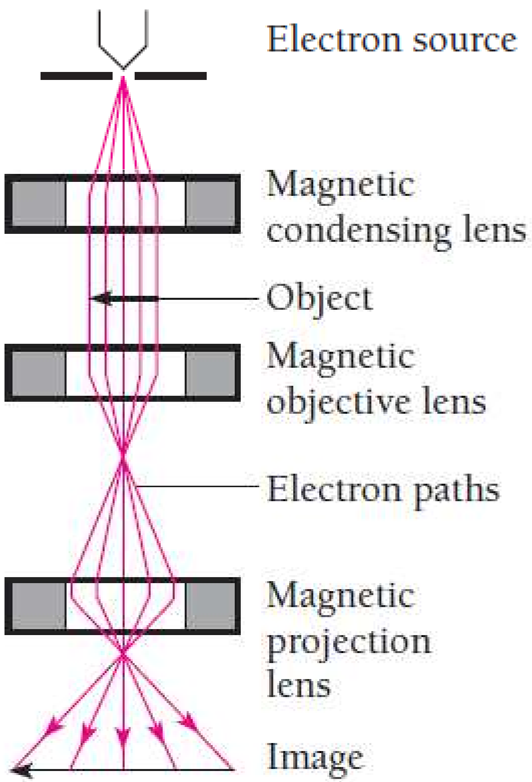


Figure 3.11 Energy levels of an electron confined to a box 0.1 nm wide.

電子顯微鏡(electron microscope)

Because the wavelengths of the fast electrons in an electron microscope are shorter than those of the light waves in an optical microscope, the electron microscope can produce sharp images at higher magnifications. The electron beam in an electron microscope is focused by magnetic fields.

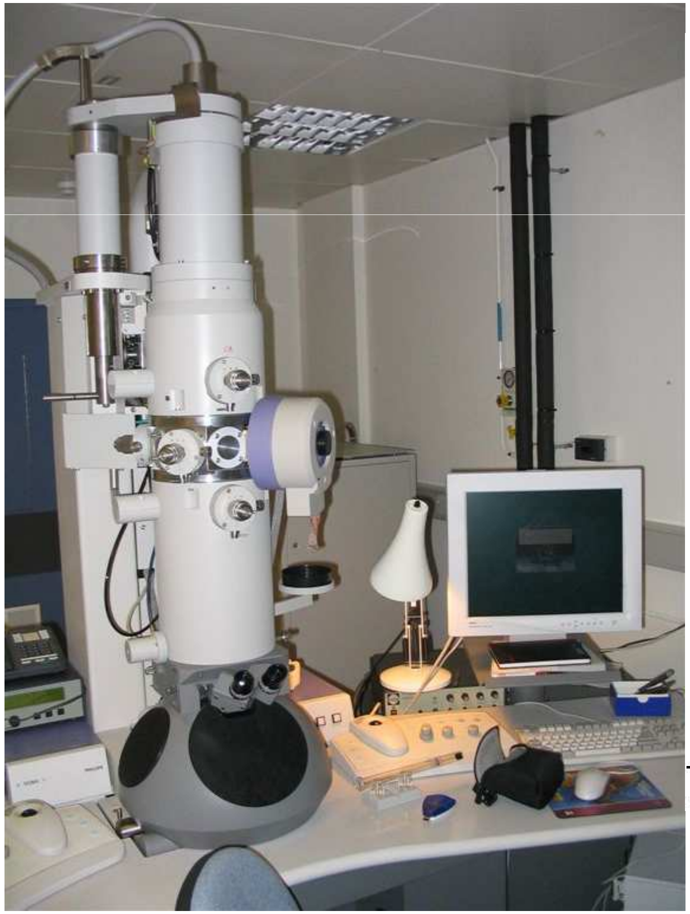


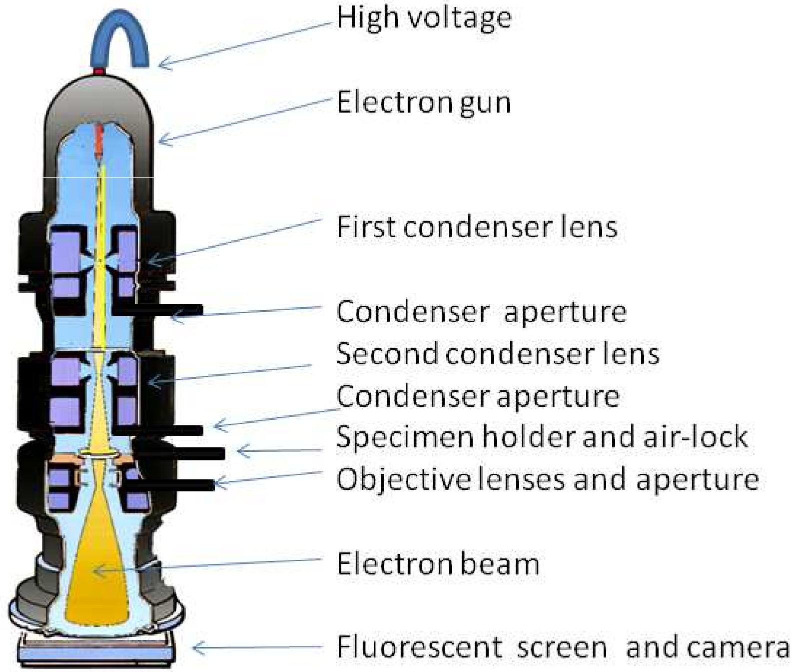
東海大學應用物理系的
穿隧式電子顯微鏡(TEM)

In the case of a good microscope that uses visible light, the maximum useful magnification is about 500.

Electron microscope has a great improvement on the 200-nm resolution of an optical microscope, and magnifications of over 1,000,000 have been achieved.

In an electron microscope, current-carrying coils produce magnetic fields that act as lenses to focus an electron beam on a specimen and then produce an enlarged image on a fluorescent screen or photographic plate (figure). To prevent the beam from being scattered and thereby blurring the image, a thin specimen is used and the entire system is evacuated.

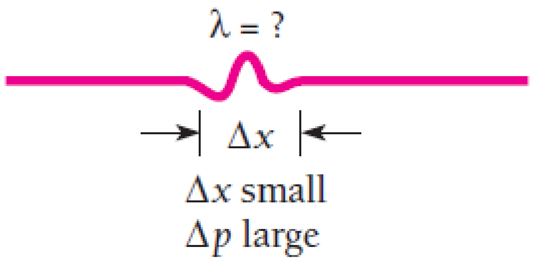


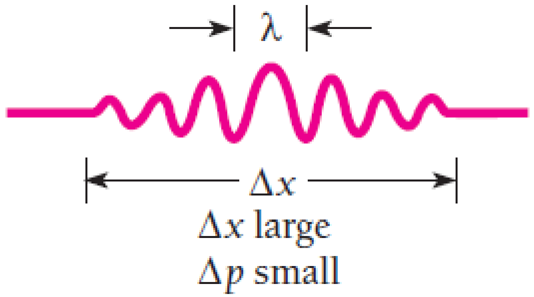


Transmission Electron Microscope

3.7 UNCERTAINTY PRINCIPLE 1

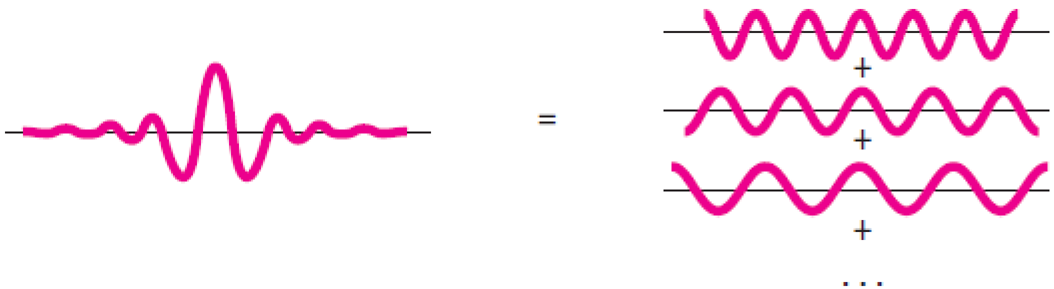
We cannot know the future because we cannot know the present





It is impossible to know both the exact position and exact momentum of an object at the same time.

$$\Psi(x) = \int_0^\infty g(k)\cos kx\, dk$$



$$\Delta x\, \Delta k \geq \tfrac{1}{2}$$

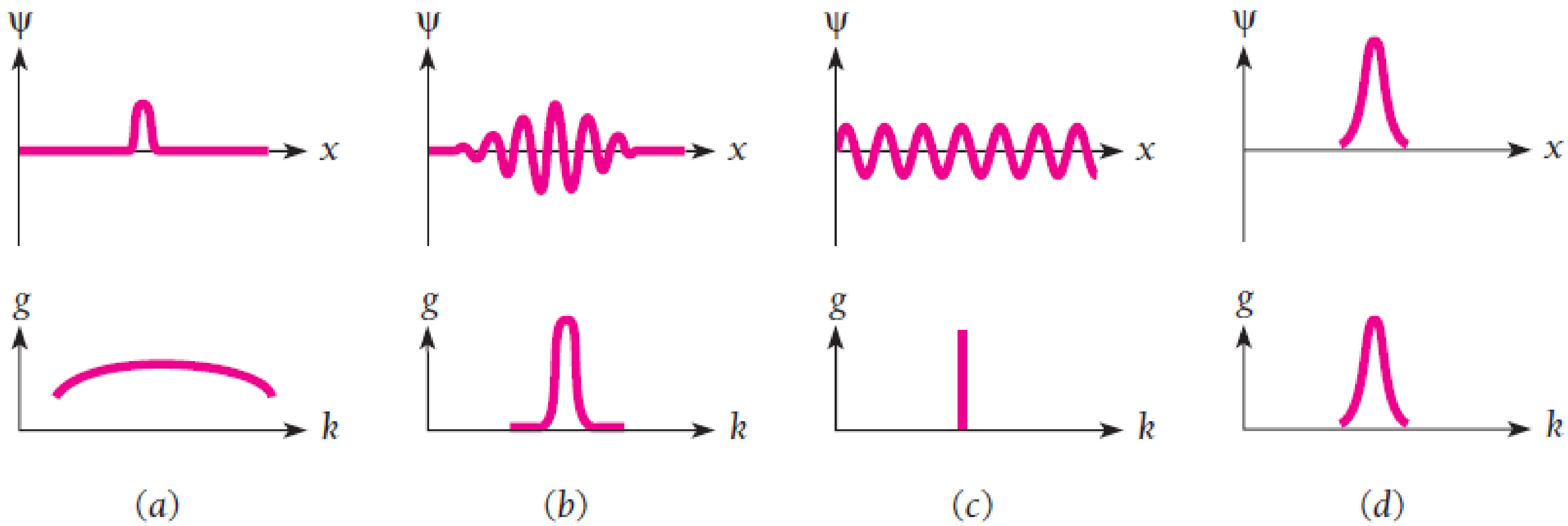


Figure 3.14 The wave functions and Fourier transforms for (*a*) a pulse, (*b*) a wave group, (*c*) a wave train, and (*d*) a Gaussian distribution. A brief disturbance needs a broader range of frequencies to describe it than a disturbance of greater duration. The Fourier transform of a Gaussian function is also a Gaussian function.

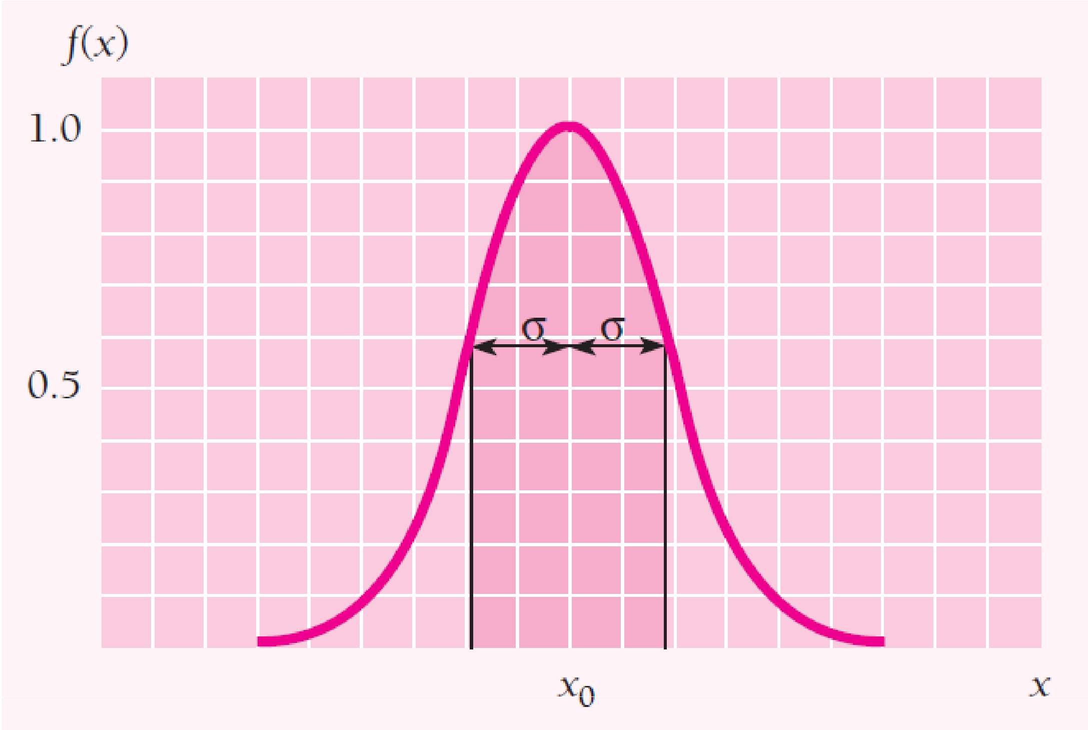
$$\Delta x \, \Delta k \geq \tfrac{1}{2}$$

Gaussian Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_0)^2/2\sigma^2}$$

$$P_{x_1x_2} = \int_{x_1}^{x_2} f(x)\, dx$$

$$P_{x_0\pm\sigma} = \int_{x_0-\sigma}^{x_0+\sigma} f(x)\, dx = 0.683$$



The probability of finding a value of x is given by the Gaussian function f(x). The mean value of x is x_0, and the total width of the curve at half its maximum value is 2.35, where is the standard deviation of the distribution. The total probability of finding a value of x within a standard deviation of x_0 is equal to the shaded area and is 68.3%. 95.4% of the measurements fall within two standard deviations of the mean value.

$$\Delta x \, \Delta k \geq \tfrac{1}{2}$$

$$\Delta x \, \Delta k \geq \tfrac{1}{2}, \; \Delta k \geq 1/(2\Delta x)$$

$$\lambda = h/p$$

$$\Delta x \, \Delta p \geq \frac{h}{4\pi}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \, \mathrm{J \cdot s}$$

$$p = \frac{hk}{2\pi}$$

$$\Delta p = \frac{h \, \Delta k}{2\pi}$$

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

Example 3.6

A measurement establishes the position of a proton with an accuracy of $\pm 1.00 \times 10^{-11}$ m. Find the uncertainty in the proton's position 1.00 s later. Assume $\boldsymbol{v} \ll c$.

Solution

Let us call the uncertainty in the proton's position Δx_0 at the time $t = 0$. The uncertainty in its momentum at this time is therefore, from Eq. (3.22),

$$\Delta p \geq \frac{\hbar}{2\Delta x_0}$$

Since $\boldsymbol{v} \ll c$, the momentum uncertainty is $\Delta p = \Delta(m\boldsymbol{v}) = m\,\Delta\boldsymbol{v}$ and the uncertainty in the proton's velocity is

$$\Delta \boldsymbol{v} = \frac{\Delta p}{m} \geq \frac{\hbar}{2m\,\Delta x_0}$$

The distance x the proton covers in the time t cannot be known more accurately than

$$\Delta x = t\,\Delta \boldsymbol{v} \geq \frac{\hbar t}{2m\,\Delta x_0}$$

Hence Δx is inversely proportional to Δx_0: the *more* we know about the proton's position at $t = 0$, the *less* we know about its later position at $t > 0$. The value of Δx at $t = 1.00$ s is

$$\Delta x \geq \frac{(1.054 \times 10^{-34}\ \mathrm{J \cdot s})(1.00\ \mathrm{s})}{(2)(1.672 \times 10^{-27}\ \mathrm{kg})(1.00 \times 10^{-11}\ \mathrm{m})}$$

$$\geq 3.15 \times 10^{3}\ \mathrm{m}$$

This is 3.15 km—nearly 2 mi! What has happened is that the original wave group has spread

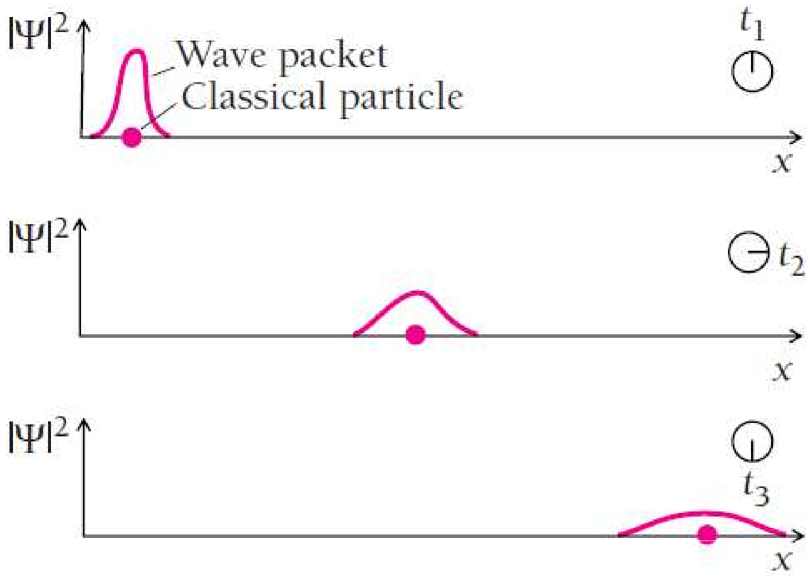


Figure 3.16 The wave packet that corresponds to a moving packet is a composite of many individual waves, as in Fig. 3.13. The phase velocities of the individual waves vary with their wave lengths. As a result, as the particle moves, the wave packet spreads out in space. The narrower the original wavepacket—that is, the more precisely we know its position at that time—the more it spreads out because it is made up of a greater span of waves with different phase velocities.

$$\Delta p \approx \frac{h}{\lambda}$$

$$\Delta x \geq \lambda$$

$$\Delta x \, \Delta p \geq h$$

$$\Delta x \, \Delta p \geq \hbar/2.$$

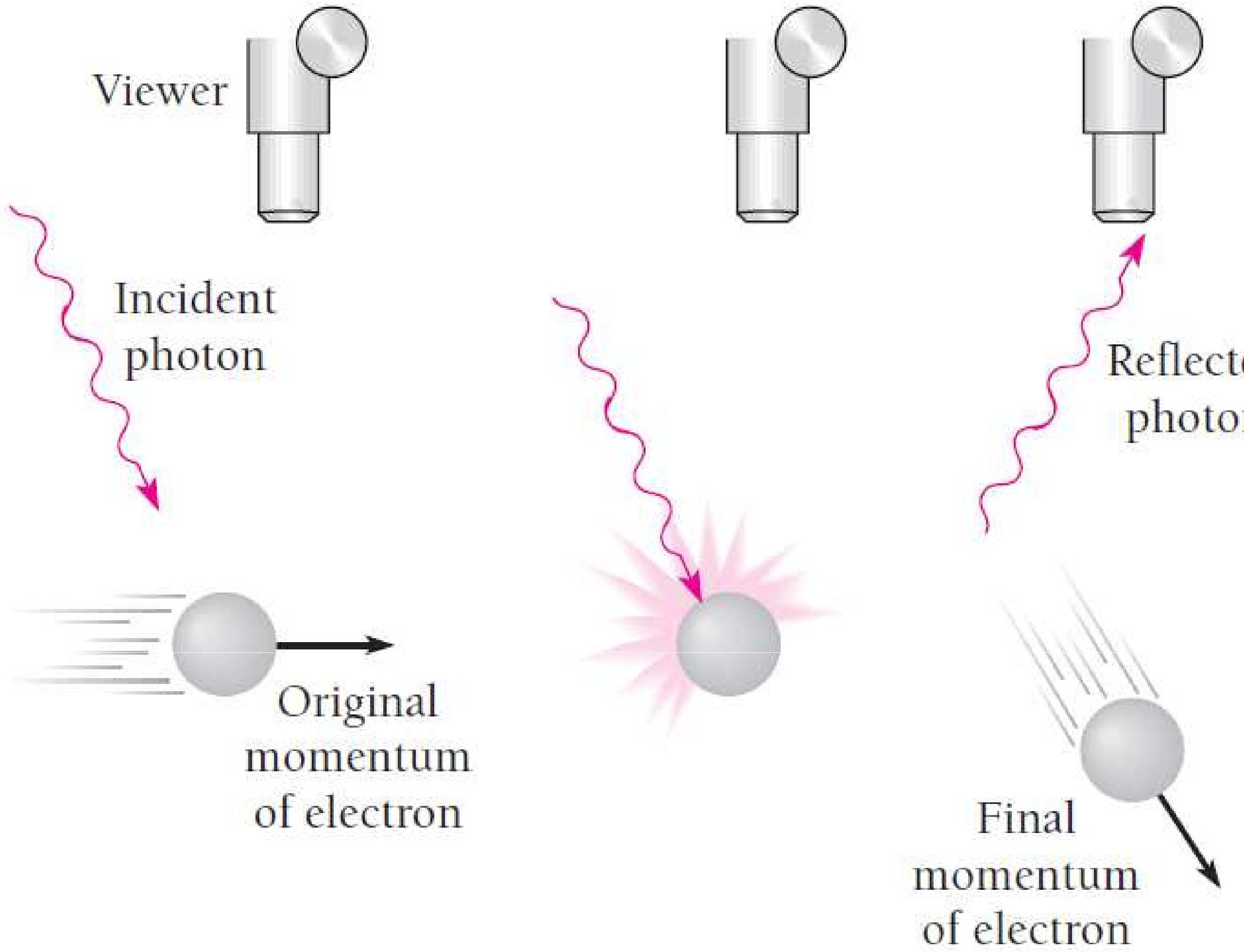


Figure 3.17 An electron cannot be observed without changing its momentum.

This indeterminacy is inherent in the nature of a moving body. 1. They show it is impossible to imagine a way around the uncertainty principle; 2, they present a view of the principle that can be appreciated in a more familiar context than that of wave groups.

3.9 APPLYING THE UNCERTAINTY PRINCIPLE

A useful tool, not just a negative statement

Example 3.7

A typical atomic nucleus is about 5.0×10^{-15} m in radius. Use the uncertainty principle to place a lower limit on the energy an electron must have if it is to be part of a nucleus.

Letting $\Delta x = 5.0 \times 10^{-5}$ m we have

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq \frac{1.054 \times 10^{-34}\ \mathrm{J \cdot s}}{(2)(5.0 \times 10^{-15}\ \mathrm{m})} \geq 1.1 \times 10^{-20}\ \mathrm{kg \cdot m/s}$$

If this is the uncertainty in a nuclear electron's momentum, the momentum p itself must be at least comparable in magnitude. An electron with such a momentum has a kinetic energy KE many times greater than its rest energy mc^2. From Eq. (1.24) we see that we can let KE $= pc$ here to a sufficient degree of accuracy. Therefore

$$\mathrm{KE} = pc \geq (1.1 \times 10^{-20}\ \mathrm{kg \cdot m/s})(3.0 \times 10^{8}\ \mathrm{m/s}) \geq 3.3 \times 10^{-12}\ \mathrm{J}$$

Since 1 eV $= 1.6 \times 10^{-19}$ J, the kinetic energy of an electron must exceed 20 MeV if it is to be inside a nucleus. Experiments show that the electrons emitted by certain unstable nuclei never have more than a small fraction of this energy, from which we conclude that nuclei cannot contain electrons. The electron an unstable nucleus may emit comes into being at the moment the nucleus decays (see Secs. 11.3 and 12.5).

Example 3.8

A hydrogen atom is 5.3×10^{-11} m in radius. Use the uncertainty principle to estimate the minimum energy an electron can have in this atom.

Solution

Here we find that with $\Delta x = 5.3 \times 10^{-11}$ m.

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq 9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s}$$

An electron whose momentum is of this order of magnitude behaves like a classical particle, and its kinetic energy is

$$\text{KE} = \frac{p^2}{2m} \geq \frac{(9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s})^2}{(2)(9.1 \times 10^{-31} \text{ kg})} \geq 5.4 \times 10^{-19} \text{ J}$$

which is 3.4 eV. The kinetic energy of an electron in the lowest energy level of a hydrogen atom is actually 13.6 eV.

Energy and Time

$$\Delta\nu \geq \frac{1}{\Delta t}$$

$$\Delta E = h\,\Delta\nu$$

$$\Delta E \geq \frac{h}{\Delta t} \qquad \text{or} \qquad \Delta E\,\Delta t \geq h$$

Uncertainties in energy and time

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}$$

Example 3.9

An "excited" atom gives up its excess energy by emitting a photon of characteristic frequency, as described in Chap. 4. The average period that elapses between the excitation of an atom and the time it radiates is 1.0×10^{-8} s. Find the inherent uncertainty in the frequency of the photon.

Solution

The photon energy is uncertain by the amount

$$\Delta E \geq \frac{\hbar}{2\Delta t} \geq \frac{1.054 \times 10^{-34}\ \mathrm{J \cdot s}}{2(1.0 \times 10^{-8}\ \mathrm{s})} \geq 5.3 \times 10^{-27}\ \mathrm{J}$$

The corresponding uncertainty in the frequency of light is

$$\Delta \nu = \frac{\Delta E}{h} \geq 8 \times 10^{6}\ \mathrm{Hz}$$

This is the irreducible limit to the accuracy with which we can determine the frequency of the radiation emitted by an atom. As a result, the radiation from a group of excited atoms does not appear with the precise frequency ν. For a photon whose frequency is, say, 5.0×10^{14} Hz, $\Delta\nu/\nu = 1.6 \times 10^{-8}$. In practice, other phenomena such as the doppler effect contribute more than this to the broadening of spectral lines.

Unit-03

Wave properties of particles

自我練習題目與例題

Sample Problem 38-5

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

$$\begin{aligned} p &= \sqrt{2mK} \\ &= \sqrt{(2)(9.11 \times 10^{-31}\ \text{kg})(120\ \text{eV})(1.60 \times 10^{-19}\ \text{J/eV})} \\ &= 5.91 \times 10^{-24}\ \text{kg}\cdot\text{m/s}. \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{h}{p} \\ &= \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{5.91 \times 10^{-24}\ \text{kg}\cdot\text{m/s}} \\ &= 1.12 \times 10^{-10}\ \text{m} = 112\ \text{pm}. \end{aligned}$$

•43 In an old-fashioned television set, electrons are accelerated through a potential difference of 25.0 kV. What is the de Broglie wavelength of such electrons? (Relativity is not needed.) SSM

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K}} = \frac{h}{\sqrt{2m_e eV}},$$

$$\lambda = \frac{h}{\sqrt{2m_e eV}} = \frac{6.626\times10^{-34}\ \mathrm{J\cdot s}}{\sqrt{2(9.109\times10^{-31}\ \mathrm{kg})(1.602\times10^{-19}\ \mathrm{C})(25.0\times10^{3}\ \mathrm{V})}}$$

$$= 7.75\times10^{-12}\ \mathrm{m} = 7.75\ \mathrm{pm}.$$

••45 What is the wavelength of (a) a photon with energy 1.00 eV, (b) an electron with energy 1.00 eV, (c) a photon of energy 1.00 GeV, and (d) an electron with energy 1.00 GeV?

45. (a) The momentum of the photon is given by $p = E/c$, where E is its energy. Its wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.00\,\text{eV}} = 1240\,\text{nm}.$$

(b) The momentum of the electron is given by $p = \sqrt{2mK}$, where K is its kinetic energy and m is its mass. Its wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}}.$$

If K is given in electron volts, then

$$\lambda = \frac{6.626\times10^{-34}\,\text{J}\cdot\text{s}}{\sqrt{2(9.109\times10^{-31}\,\text{kg})(1.602\times10^{-19}\,\text{J/eV})K}} = \frac{1.226\times10^{-9}\,\text{m}\cdot\text{eV}^{1/2}}{\sqrt{K}} = \frac{1.226\,\text{nm}\cdot\text{eV}^{1/2}}{\sqrt{K}}.$$

For $K = 1.00$ eV, we have

$$\lambda = \frac{1.226\,\text{nm}\cdot\text{eV}^{1/2}}{\sqrt{1.00\,\text{eV}}} = 1.23\text{ nm}.$$

(c) For the photon,

$$\lambda = \frac{hc}{E} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.00\times10^{9}\,\text{eV}} = 1.24\times10^{-6}\,\text{nm} = 1.24\ \text{fm}.$$

(d) Relativity theory must be used to calculate the wavelength for the electron. According to Eq. 38-51, the momentum p and kinetic energy K are related by

$$(pc)^2 = K^2 + 2Kmc^2.$$

Thus,

$$pc = \sqrt{K^2 + 2Kmc^2} = \sqrt{\left(1.00\times10^{9}\,\text{eV}\right)^2 + 2\left(1.00\times10^{9}\,\text{eV}\right)\left(0.511\times10^{6}\,\text{eV}\right)}$$
$$= 1.00\times10^{9}\,\text{eV}.$$

The wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.00\times10^{9}\,\text{eV}} = 1.24\times10^{-6}\,\text{nm} = 1.24\ \text{fm}.$$

••47 Singly charged sodium ions are accelerated through a potential difference of 300 V. (a) What is the momentum acquired by such an ion? (b) What is its de Broglie wavelength? **SSM WWW**

47. (a) The kinetic energy acquired is $K = qV$, where q is the charge on an ion and V is the accelerating potential. Thus

$$K = (1.60 \times 10^{-19}\ \text{C})(300\ \text{V}) = 4.80 \times 10^{-17}\ \text{J}.$$

The mass of a single sodium atom is, from Appendix F,

$$m = (22.9898\ \text{g/mol})/(6.02 \times 10^{23}\ \text{atom/mol}) = 3.819 \times 10^{-23}\ \text{g} = 3.819 \times 10^{-26}\ \text{kg}.$$

Thus, the momentum of an ion is

$$p = \sqrt{2mK} = \sqrt{2\left(3.819\times10^{-26}\ \text{kg}\right)\left(4.80\times10^{-17}\ \text{J}\right)} = 1.91\times10^{-21}\ \text{kg}\cdot\text{m/s}.$$

(b) The de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{1.91\times10^{-21}\ \text{kg}\cdot\text{m/s}} = 3.46\times10^{-13}\ \text{m}.$$

••49 The wavelength of the yellow spectral emission line of sodium is 590 nm. At what kinetic energy would an electron have that wavelength as its de Broglie wavelength? **SSM**

49. If K is given in electron volts, then

$$\lambda=\frac{h}{p}=\frac{h}{\sqrt{2mK}}=\frac{6.626\times10^{-34}\ \mathrm{J\cdot s}}{\sqrt{2(9.109\times10^{-31}\ \mathrm{kg})(1.602\times10^{-19}\ \mathrm{J/eV})K}}=\frac{1.226\times10^{-9}\ \mathrm{m\cdot eV^{1/2}}}{\sqrt{K}}$$

$$=\frac{1.226\mathrm{nm\cdot eV^{1/2}}}{\sqrt{K}},$$

where K is the kinetic energy. Thus,

$$K=\left(\frac{1.226\,\mathrm{nm\cdot eV^{1/2}}}{\lambda}\right)^2=\left(\frac{1.226\,\mathrm{nm\cdot eV^{1/2}}}{590\,\mathrm{nm}}\right)^2=4.32\times10^{-6}\ \mathrm{eV}.$$

Certain ocean waves travel with a phase velocity $v_{\text{phase}} = \sqrt{g\lambda/2\pi}$, where g is the acceleration due to gravity. What is the group velocity of a "wave packet" of these waves?

With $k = 2\pi/\lambda$, we can write the phase velocity as a function of k as

$$v_{\text{phase}} = \sqrt{g/k}$$

But with $v_{\text{phase}} = \omega/k$, we have $\omega/k = \sqrt{g/k}$, so $\omega = \sqrt{gk}$ and Eq. 4.28 gives

$$v_{\text{group}} = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

Note that the group speed of the wave packet increases as the wavelength increases.

Estimate the kinetic energy of an electron confined within a nucleus of size 1.0×10^{-14} m by using the uncertainty principle.

Solution Taking Δx to be the half-width of the confinement length in the equation $\Delta p_x \geq \frac{\hbar}{2\,\Delta x}$, we have

$$\Delta p_x \geq \frac{6.58 \times 10^{-16}\ \mathrm{eV \cdot s}}{1.0 \times 10^{-14}\ \mathrm{m}} \times \frac{3.00 \times 10^{8}\ \mathrm{m/s}}{c}$$

or

$$\Delta p_x \geq 2.0 \times 10^{7}\ \frac{\mathrm{eV}}{c}$$

This means that measurements of the component of momentum of electrons trapped inside a nucleus would range from less than $-20\ \mathrm{MeV}/c$ to greater than $+20\ \mathrm{MeV}/c$ and that some electrons would have momentum at least as large as $20\ \mathrm{MeV}/c$. Because this appears to be a large momentum, to be safe we calculate the electron's energy relativistically.

$$\begin{aligned} E^2 &= p^2c^2 + (m_e c^2)^2 \\ &= (20\ \mathrm{MeV}/c)^2 c^2 + (0.511\ \mathrm{MeV})^2 \\ &= 400(\mathrm{MeV})^2 \end{aligned}$$

or

$$E \geq 20\ \mathrm{MeV}$$

Finally, the kinetic energy of an intranuclear electron is

$$K = E - m_e c^2 \geq 19.5\ \mathrm{MeV}$$

Since electrons emitted in radioactive decay of the nucleus (beta decay) have energies much less than 19.5 MeV (about 1 MeV or less) and it is known that no other mechanism could carry off an intranuclear electron's energy during the decay process, we conclude that electrons observed in beta decay do not come from within the nucleus but are actually created at the instant of decay.

Although an excited atom can radiate at any time from $t = 0$ to $t = \infty$, the average time after excitation at which a group of atoms radiates is called the **lifetime,** τ, of a particular excited state. (a) If $\tau = 1.0 \times 10^{-8}$ s (a typical value), use the uncertainty principle to compute the line width Δf of light emitted by the decay of this excited state.

(b) If the wavelength of the spectral line involved in this process is 500 nm, find the fractional broadening $\Delta f / f$.

Solution We use $\Delta E\,\Delta t \approx \hbar/2$, where ΔE is the uncertainty in energy of the excited state, and $\Delta t = 1.0 \times 10^{-8}$ s is the average time available to measure the excited state. Thus,

$$\Delta E \approx \hbar/2\,\Delta t = \hbar/(2.0 \times 10^{-8}\ \text{s})$$

Since ΔE is also the uncertainty in energy of a photon emitted when the excited state decays, and $\Delta E = h\Delta f$ for a photon,

$$h\,\Delta f = \hbar/(2.0 \times 10^{-8}\ \text{s})$$

or

$$\Delta f = \frac{1}{4\pi \times 10^{-8}\ \text{s}} = 8.0 \times 10^{6}\ \text{Hz}$$

Solution First, we find the center frequency of this line as follows:

$$f_0 = \frac{c}{\lambda} = \frac{3.0 \times 10^{8}\ \text{m/s}}{500 \times 10^{-9}\ \text{m}} = 6.0 \times 10^{14}\ \text{Hz}$$

Hence,

$$\frac{\Delta f}{f_0} = \frac{8.0 \times 10^{6}\ \text{Hz}}{6.0 \times 10^{14}\ \text{Hz}} = 1.3 \times 10^{-8}$$

This narrow natural line width can be seen with a sensitive interferometer. Usually, however, temperature and pressure effects overshadow the natural line width and broaden the line through mechanisms associated with the Doppler effect and atomic collisions.

An electron is trapped in a one-dimensional region of length 1.00×10^{-10} m (a typical atomic diameter). (*a*) Find the energies of the ground state and first two excited states. (*b*) How much energy must be supplied to excite the electron from the ground state to the second excited state?

(*a*) The basic quantity of energy needed for this calculation is

$$E_0 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8mc^2L^2}$$

$$= \frac{(1240\,\text{eV}\cdot\text{nm})^2}{8(511,000\,\text{eV})(0.100\,\text{nm})^2} = 37.6\,\text{eV}$$

With $E_n = n^2 E_0$, we can find the energy of the states:

$$n = 1: \quad E_1 = E_0 = 37.6\,\text{eV}$$

$$n = 2: \quad E_2 = 4E_0 = 150.4\,\text{eV}$$

$$n = 3: \quad E_3 = 9E_0 = 338.4\,\text{eV}$$

(*b*) The energy difference between the ground state and the second excited state is

$$\Delta E = E_3 - E_1 = 338.4\,\text{eV} - 37.6\,\text{eV} = 300.8\,\text{eV}$$

5-17. Two harmonic waves travel simultaneously along a long wire. Their wave functions are $y_1 = 0.002\cos(8.0x - 400t)$ and $y_2 = 0.002\cos(7.6x - 380t)$, where y and x are in meters and t in seconds. (*a*) Write the wave function for the resultant wave in the form of Equation 5-15. (*b*) What is the phase velocity of the resultant wave? (*c*) What is the group velocity? (*d*) Calculate the range Δx between successive zeros of the group and relate it to Δk.

5-1. (a) 2.1×10^{-23} m (b) 2.1×10^{-21} m/y

5-5. 0.0276 nm

5-9. (a) 0.445 fm (b) 6.18×10^{-3} fm

5-13. $\lambda = 0.523$ nm; $E_k = 3.0 \times 10^{-3}$ eV

5-17. (a) See *SSM* (b) 50 m/s (c) 50 m/s (d) $\Delta x = 5\pi$ m; $\Delta k = 0.4$ m^{-1}

5-21. 3.2×10^{-5} s

5-25. (a) $A^2\,dx$ (b) $0.61A^2\,dx$ (c) $0.14A^2\,dx$ (d) $x = 0$

5-29. 1.99×10^{-21} eV

5-33. (a) 5.3×10^{-10} (b) 1.32×10^{-7} eV

5-37. See *SSM*

5-41. (a) See *SSM* (b) See *SSM*

5-45. (a) 1840 MeV (b) 2.02 fm (c) 1.22 fm (d) 0.76 fm

5-49. (a) 0.243 nm (b) 0.511 MeV (c) 0.511 MeV/c (d) 2.43×10^{-3} nm

5-53. 1.2×10^{-6} eV, 1.2 eV

5-25. The wave function describing a state of an electron confined to move along the x axis is given at time zero by

$$\Psi(x, 0) = Ae^{-x^2/4\sigma^2}$$

Find the probability of finding the electron in a region dx centered at (*a*) $x = 0$, (*b*) $x = \sigma$, and (*c*) $x = 2\sigma$. (*d*) Where is the electron most likely to be found?

5-1. (a) 2.1×10^{-23} m (b) 2.1×10^{-21} m/y

5-5. 0.0276 nm

5-9. (a) 0.445 fm (b) 6.18×10^{-3} fm

5-13. $\lambda = 0.523$ nm; $E_k = 3.0 \times 10^{-3}$ eV

5-17. (a) See *SSM* (b) 50 m/s (c) 50 m/s (d) $\Delta x = 5\pi$ m; $\Delta k = 0.4\ \text{m}^{-1}$

5-21. 3.2×10^{-5} s

5-25. (a) $A^2\,dx$ (b) $0.61A^2\,dx$ (c) $0.14A^2\,dx$ (d) $x = 0$

5-29. 1.99×10^{-21} eV

5-33. (a) 5.3×10^{-10} (b) 1.32×10^{-7} eV

5-37. See *SSM*

5-41. (a) See *SSM* (b) See *SSM*

5-45. (a) 1840 MeV (b) 2.02 fm (c) 1.22 fm (d) 0.76 fm

5-49. (a) 0.243 nm (b) 0.511 MeV (c) 0.511 MeV/c (d) 2.43×10^{-3} nm

5-53. 1.2×10^{-6} eV, 1.2 eV

5-29. ^{222}Rn decays by the emission of an α particle with a lifetime of 3.823 days. The kinetic energy of the α particle is measured to be 5.490 MeV. What is the uncertainty in this energy? Describe in one sentence how the finite lifetime of the excited state of the radon nucleus translates into an energy uncertainty for the emitted α particle.

5-1.	(a) 2.1×10^{-23} m (b) 2.1×10^{-21} m/y
5-5.	0.0276 nm
5-9.	(a) 0.445 fm (b) 6.18×10^{-3} fm
5-13.	$\lambda = 0.523$ nm; $E_k = 3.0 \times 10^{-3}$ eV
5-17.	(a) See *SSM* (b) 50 m/s (c) 50 m/s (d) $\Delta x = 5\pi$ m; $\Delta k = 0.4$ m^{-1}
5-21.	3.2×10^{-5} s
5-25.	(a) $A^2\,dx$ (b) $0.61A^2\,dx$ (c) $0.14A^2\,dx$ (d) $x = 0$
5-29.	1.99×10^{-21} eV
5-33.	(a) 5.3×10^{-10} (b) 1.32×10^{-7} eV
5-37.	See *SSM*

5-33. The decay of excited states in atoms and nuclei often leave the system in another, albeit lower-energy, excited state. (*a*) One example is the decay between two excited states of the nucleus of ^{48}Ti. The upper state has a lifetime of 1.4 ps, the lower state 3.0 ps. What is the fractional uncertainty $\Delta E/E$ in the energy of 1.3117-MeV gamma rays connecting the two states? (*a*) Another example is the H_α line of the hydrogen Balmer series. In this case the lifetime of both states is about the same, 10^{-8} s. What is the uncertainty in the energy of the H_α photon?

5-49. An electron and a positron are moving toward each other with equal speeds of 3×10^6 m/s. The two particles annihilate each other and produce two photons of equal energy. (*a*) What were the de Broglie wavelengths of the electron and positron? Find the (*b*) energy, (*c*) momentum, and (*d*) wavelength of each photon.

5-1. (a) 2.1×10^{-23} m (b) 2.1×10^{-21} m/y

5-5. 0.0276 nm

5-9. (a) 0.445 fm (b) 6.18×10^{-3} fm

5-13. $\lambda = 0.523$ nm; $E_k = 3.0 \times 10^{-3}$ eV

5-17. (a) See *SSM* (b) 50 m/s (c) 50 m/s (d) $\Delta x = 5\pi$ m; $\Delta k = 0.4$ m^{-1}

5-21. 3.2×10^{-5} s

5-25. (a) $A^2\,dx$ (b) $0.61A^2\,dx$ (c) $0.14A^2\,dx$ (d) $x = 0$

5-29. 1.99×10^{-21} eV

5-33. (a) 5.3×10^{-10} (b) 1.32×10^{-7} eV

5-37. See *SSM*

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5-45. (a) 1840 MeV (b) 2.02 fm (c) 1.22 fm (d) 0.76 fm

5-49. (a) 0.243 nm (b) 0.511 MeV (c) 0.511 MeV/c (d) 2.43×10^{-3} nm

5-53. 1.2×10^{-6} eV, 1.2 eV