The Derivative
導數

§3-1 導數與變化率

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

導數的定義：

一函數 f 的導數 f' 定義為

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ ，對於所有 x 均存在。

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

微分記號：

If $y=f(x)$ ，$\frac{dy}{dx}=f'(x)$

§3-2 基本微分規則

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The function notation $f'(x)$

The differential notation $\frac{dy}{dx}$

The operator notation $Df(x)$

Theorem 1：Derivative of a Constant（常數的微分）：

If $f(x)=c$ (a constant) for all x, then $f'(x)=0$ for all x. That is,

$$\frac{dc}{dx}=Dc=0$$

<證明>：因為 $f(x+h)=f(x)=c$

所以 $f'(a)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0\ldots\#$

$Dx = 1$
$Dx^2 = 2x$
$Dx^3 = 3x^2$
$Dx^4 = 4x^3$
$Dx^{-1} = -x^{-2}$
$Dx^{-2} = -2x^{-3}$
$Dx^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$
$Dx^{-\frac{1}{2}} = -\frac{1}{2}x^{-\frac{3}{2}}$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2\cdot 1}a^{n-2}b^2 + \ldots + \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 3\cdot 2\cdot 1}a^{n-k}b^k + \ldots + nab^{n-1} + b^n$$

$n=2 \quad (a+b)^2 = a^2 + 2ab + b^2$

$n=3 \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{aligned}(x+h)^5 &= x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 \\ &= x^5 + 5x^4h + h^2\left(10x^3 + 10x^2h + 5xh^2 + h^3\right) \\ &= x^5 + 5x^4h + h^2\cdot P(h)\end{aligned}$$

$$(x+h)^n = x^n + nx^{n-1}h + h^2P(h)$$

Theorem 2：Power rule for a positive integer n

If n is a position integer and $f(x) = x^n$, then $f'(x) = nx^{n-1}$

<證明>：

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n-(x)^n}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h + h^2P(h)}{h} \\ &= \lim_{h\to 0}\left[nx^{n-1} + h\cdot P(h)\right] \\ &= nx^{n-1} + 0\cdot P(0)\end{aligned}$$

$f'(x) = nx^{n-1}$ …#

Theorem 3：Derivative of a Linear Combination（線性組合的導數）：

If f and g are differentiable function and a and b are fixed real number,

then $D\left[af(x)+bg(x)\right] = aDf(x) + bDg(x)$

With $u = f(x)$ and $v = g(x)$, this takes the form

$$\frac{d\left(au+bv\right)}{dx}=a\frac{du}{dx}+b\frac{dv}{dx}$$

<證明>： $D\left[af(x)+bg(x)\right]=\lim\limits_{h\to 0}\dfrac{\left[af(x+h)+bg(x+h)\right]-\left[af(x)+bg(x)\right]}{h}$

$$=a\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)+b\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

$$=aDf(x)+bDg(x)\ldots\#$$

Theorem 4：The Product Rule

If f and g are differential at x, then fg is differentiable at x, and

$$D\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

With $u=f(x)$ and $v=g(x)$, this **product rule** takes the form

$$\frac{d\left(uv\right)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

When it is clear what the independent variable is, we can make the product rule even briefer:

$$(uv)'=u'v+uv'$$

<證明>： $D\left[f(x)g(x)\right]=\lim\limits_{h\to 0}\dfrac{f(x+h)g(x+h)-f(x)g(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0}g(x+h)\right)+\left(f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

$$=f'(x)g(x)+f(x)g'(x)\ldots\#$$

The Reciprocal Rule

If f is differentiable at x and $f(x)\neq 0$, then

$$D\frac{1}{f(x)}=-\frac{f'(x)}{\left[f(x)\right]^2}$$

With $u=f(x)$, the reciprocal rule takes the form

$$\frac{d}{dx}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\cdot\frac{du}{dx}$$

If there can be no doubt what the independent variable is, we can write

$$\left(\frac{1}{u}\right)'=-\frac{u'}{u^2}$$

<證明>：$D\frac{1}{f(x)}=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{f(x+h)}-\frac{1}{f(x)}\right]$

$$=\lim_{h\to 0}\frac{f(x)-f(x+h)}{hf(x+h)f(x)}$$

$$=-\left(\lim_{h\to 0}\frac{1}{f(x+h)f(x)}\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$

$$=-\frac{f'(x)}{[f(x)]^2}\ldots\#$$

Theorem 5：Power rule for a negative Integer n

If n is a negative integer, then $Dx^n=nx^{n-1}$

<證明>：令 $m=-n$

$$Dx^n=D\frac{1}{x^m}=-\frac{D(x^m)}{(x^m)^2}$$

$$=-\frac{mx^{m-1}}{x^{2m}}=(-m)x^{(-m)-1}=nx^{n-1}\ldots\#$$

Theorem 6：The Quotient Rule

If f and g are differentiable at x and $g(x)\neq 0$, then $\frac{f}{g}$ is differentiable at x and

$$D\frac{f(x)}{g(x)}=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

With $u=f(x)$ and $v=g(x)$, this rule takes the form

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

If it is clear what the independent variable is, we can write the quotient rule as

$$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

<證明>：$\frac{f(x)}{g(x)}=f(x)\cdot\frac{1}{g(x)}$

$$D\frac{f(x)}{g(x)}=[Df(x)]\frac{1}{g(x)}+f(x)\cdot D\frac{1}{g(x)}$$

$$= \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \ldots \#$$

mengwen的筆記

§3-3 The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$y = g(u)$ and $u = f(x)$

➲ $y = h(x) = g[f(x)]$

$$\frac{du}{dx} = f'(x)$$

$$\frac{dy}{dx} = h'(x)$$

$$\frac{dy}{du} = g'(u) = g'[f(x)]$$

➲ $h'(x) = g'[f(x)] \cdot f'(x)$

The version of the chain rule gives the derivation of the composition $h = g \circ f$ of two functions g and f in terms of their derivations.

Theorem 1：The Chain Rule

Suppose that f is differentiable at x and that g is differentiable at $f(x)$. Then the composition $h = g \circ f$ defined by $h(x) = g[f(x)]$ is differentiable at x, and its derivative is

$h'(x) = D[g[f(x)]] = g'[f(x)] \cdot f'(x)$

The Chain Rule 的證明：

<證明>： $y = g(u)$ and $u = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g[f(x+\Delta x)] - g[f(x)]}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}=\left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right)=\frac{dy}{du}\frac{du}{dx}$$

$u=f(x)$ and $f'(x)=\dfrac{du}{dx}$

$$D_x g(u)=g'(u)\frac{du}{dx}$$

If $g(u)=u^n$

$$g'(u)=nu^{n-1}$$

$$D_x u^n=nx^{n-1}\frac{du}{dx}$$

If $u=f(x)$

$$D_x\left[f(x)\right]^n=n\left[f(x)\right]^{n-1}f'(x)$$

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§3-4

$$D_x u^n=nu^{n-1}\frac{du}{dx}$$

$$u^{\frac{p}{q}}=\sqrt[q]{u^p}=\left(\sqrt[q]{u}\right)^p$$

$$y=x^{\frac{p}{q}}$$

$$y^q=x^p$$

$$D_x\left(y^q\right)=D_x\left(x^p\right)$$

$$qy^{q-1}\frac{dy}{dx}=px^{p-1}$$

$$\frac{dy}{dx}=\frac{px^{p-1}}{qy^{q-1}}=\frac{p}{q}x^{p-1}y^{1-q}=\frac{p}{q}x^{p-1}\left(x^{\frac{p}{q}}\right)^{1-q}=\frac{p}{q}x^{p-1}x^{\frac{p}{q}}x^{-p}=\frac{p}{q}x^{\left(\frac{p}{q}-1\right)}$$

$$D_x x^r=rx^{r-1}$$

$$y=u^r$$

$$\frac{dy}{du}=ru^{r-1}$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=ru^{r-1}\frac{du}{dx}$$

$$D_x u^r=ru^{r-1}\frac{du}{dx}$$

Theorem 1：Generalized Power Rule

If r is a rational number, then

$D_x\left[f(x)\right]^r = r\left[f(x)\right]^{r-1} f'(x)$

Wherever the function f is differentiable and the right-hand side is defined.

Vertical Tangent Line

The curve $y = f(x)$ has a vertical tangent line at the point $(a, f(a))$ provided that f is continuous at a and that $\left|f'(x)\right| \to +\infty$ as $x \to a$.

§3-5

【待補】

§3-6

【待補】

§3-7 三角函數的微分

There are π radians in 180 degrees.

Radians	Degrees
0	0
$\pi/4$	45
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360
4π	720

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$$

Addition formulas（加法公式）：

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

Theorem 1：Derivatives of Sines and Cosines

The functions $f(x) = \sin x$ and $g(x) = \cos x$ are differentiable for all x and

$D\sin x = \cos x$

$D\cos x = -\sin x$

<證明>：

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\left[(\cos x)\left(\frac{\sin h}{h}\right)-(\sin x)\left(\frac{1-\cos h}{h}\right)\right] \\ &= \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)-(\sin x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right) \\ &= \cos x(1)-(\sin x)(0) \\ &= \cos x \end{aligned}$$

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad$ $\cot x = \dfrac{\cos x}{\sin x}$

$\sec x = \dfrac{1}{\cos x}$ $\qquad$ $\csc x = \dfrac{1}{\sin x}$

Theorem 2：Derivatives of Trigonometric Functions

The functions $f(x) = \tan x$, $g(x) = \cot x$, $p(x) = \sec x$, and $q(x) = \csc x$ are differentiable wherever they are defined, and

$D\tan x = \sec^2 x$

$D\cot x = -\csc^2 x$

$D\sec x = \sec x \tan x$

$D\csc x = -\csc x \cot x$

<證明 1>： $D\tan x = D\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{(D\sin x)(\cos x)-(\sin x)(D\cos x)}{(\cos x)^2}$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x \ldots \#$$

<證明 2>： $D\cot x = D\left(\frac{\cos x}{\sin x}\right) = \frac{(D\cos x)(\sin x) - (\cos x)(D\sin x)}{(\sin x)^2}$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\csc^2 x \ldots \#$$

<證明 3>： $D\sec x = D\left(\frac{1}{\cos x}\right) = -\frac{D\cos x}{(\cos x)^2}$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x \ldots \#$$

<證明 4>： $D\csc x = D\frac{1}{\sin x} = -\frac{D\sin x}{(\sin x)^2}$

$$= \frac{\cos x}{\sin^2 x} = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\csc x \cot x \ldots \#$$

【NOTE】 一般習慣 $(\sin x)^2 = \sin^2 x$

$(\sin x)^3 = \sin^3 x$

但 $\sin x^2 = \sin\left(x^2\right) \neq (\sin x)^2$

還有 $(\sin x)^{-1} = \frac{1}{\sin x} \neq \sin^{-1} x$

$\sin x = a \quad \Rightarrow \quad x = \sin^{-1} a$
