The Integral
積分

§5-1 積分

§5-2 反導數與積分植

Antiderivative 反導數

An antiderivative of the function f is a function F such that

$$F'(x) = f(x)$$

Wherever $f(x)$ is defined.

Theorem 1：The Most General Antiderivative

If $F'(x) = f(x)$ at each point of the open interval I, then every antiderivative G of f on I has the form

$$G(x) = F(x) + C$$

Where C is a constant

Theorem 2：Some Integral Formulas

$$\int x^k dx = \frac{x^{k+1}}{k+1} + C \quad (\text{if } k \neq -1)$$

$$\int \cos kx dx = \frac{1}{k}\sin kx + C$$

$$\int \sin kx dx = -\frac{1}{k}\cos kx + C$$

$$\int \sec^2 kx dx = \frac{1}{k}\tan kx + C$$

$$\int \csc^2 kx dx = -\frac{1}{k}\cot kx + C$$

$$\int \sec kx \tan kx dx = \frac{1}{k}\sec kx + C$$

$$\int \csc kx \cot kx dx = -\frac{1}{k}\csc kx + C$$

$$\frac{dy}{dx}=f(x) \quad ➲ \quad y(x)=\int f(x)dx+C$$

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§5-3 面積元素計算

Summation Notation（求和記號）

$$\sum_{i=1}^{n} a_i = a_1+a_2+a_3+...+a_n$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i+b_i)=\left(\sum_{i=1}^{n} a_i\right)+\left(\sum_{i=1}^{n} b_i\right)$$

If $a_i=a$（常數）

則 $\sum_{i=1}^{n} a_i = \underbrace{a+a+a+...+a}_{\text{有}n\text{項}}$

所以 $\sum_{i=1}^{n} a = na$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n}(a+b_i)=\left(\underbrace{a+a+a+...+a}_{\text{有}n\text{項}}\right)+\left(\sum_{i=1}^{n} b_i\right)$$

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}$$

面積求和：

一個函數 $y=f(x)$ 在 $[a,b]$ 區間所圍出來的面積，如圖

（補圖）

將$[a,b]$區間分成固定的n等分，每一等分大小為$\Delta x = \dfrac{b-a}{n}$

所以每一個區塊的下緣估計值為 $\underline{A}_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

上緣估計值為 $\overline{A}_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$\underline{A}_n \le A \le \overline{A}_n$

$\sum_{i=1}^{n} f(x_{i-1})\Delta x \le A \le \sum_{i=1}^{n} f(x_i)\Delta x$

關於π：
畫一個半徑為 1（$r=1$）的單位圓。
對這個圓，分別畫內接等邊形和外切等邊形。
假設內接等邊形面積為P_n，外切等邊形面積為Q_n。

（補圖）

我們知道圓的面積為πr^2。因為是單位圓，所以面積為π。

令α_n為等邊形的半角，如圖所畫。亦即$\alpha_n = \dfrac{360^0}{2n} = \dfrac{180^0}{n}$

可以求出圓的面積下限為$\underline{A}_n = a(P_n) = n \cdot 2 \cdot \dfrac{1}{2}\sin\alpha_n \cos\alpha_n = \dfrac{n}{2}\sin 2\alpha_n = \dfrac{n}{2}\sin\left(\dfrac{360^0}{n}\right)$

面積上限為$\overline{A}_n = a(Q_n) = n \cdot 2 \cdot \dfrac{1}{2}\tan\alpha_n = n\tan\left(\dfrac{180^0}{n}\right)$

➲ $\underline{A}_n \le \pi \le \overline{A}_n$

n	$\underline{A}_n = a(P_n)$	$\overline{A}_n = a(Q_n)$
6	2.598076	3.464102
12	3.000000	3.215390
24	3.105829	3.159660
48	3.132629	3.146086
96	3.139350	3.142715
180	3.140955	3.141912
360	3.141433	3.141672
720	3.141553	3.141613
1440	3.141583	3.141598
2880	3.141590	3.141594
5760	3.141592	3.141593

§5-4 Riemann Sums（黎曼和）與積分

Riemann Sum

Let f be a function defined on the interval $[a,b]$. If P is a partition of $[a,b]$ and S is a selection for P, then the Rimann sum for f determined by P and S is

$$R=\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

We also say that this Remamm sum is associated with the partition P.

The Definite Integral

The definite integral of the function f from a to b is the number

$$I=\lim_{|P|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Provided that this limit exists, in which case we say that f is integrable on $[a,b]$. Equation means that, for each number $\varepsilon>0$, there exists a number $\delta>0$ such that

$$\left|I-\sum_{i=1}^{n} f(x_i^*)\Delta x_i\right|<\varepsilon$$

For every Rimann sum associated with any partition P of $[a,b]$ for which $|P|<\delta$.

Theorem 1：Existence of the Integral

If the function f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Theorem 2：The Integral as a Limit of a Sequence

The function f is integrable on $[a,b]$ with integral I if and only if

$$\lim_{n\to\infty} R_n = 1$$

For ever sequence $\{R_n\}_1^\infty$ of Riemann sum associated with a sequence of partitions $\{P_n\}_1^\infty$ of $[a,b]$ such that $|P_n|\to 0$ as $n\to+\infty$.

§5-5 Evaluation of Integrals

Theorem 1：Evaluation of Integrals

If G is an sntidervative of the continuous function f on the interval $[a,b]$, then

$$\int_a^b f(x)dx = G(b) - G(a)$$

$$\int_a^b f(x)dx = [G(x)]_a^b$$
$$= G(b) - G(a)$$

Integral of a Constant

$$\int_a^b cdx = c(b-a)$$

Constant Multiple Property

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

Interval Union Property

If $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Comparison Property

If $m \le f(x) \le M$ for all x in $[a,b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

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§5-6 平均值與微積分基礎

平均值：$\bar{a} = \dfrac{a_1 + a_2 + a_3 + \ldots + a_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} a_i$

Average Value of a Function

Support that the function f is integrable on $[a,b]$. Then the average value $\bar{y}$ of $y = f(x)$ on $[a,b]$ is

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 1：Average Value Theorem

If f is continuous on $[a,b]$, then

$$f(\bar{x})=\frac{1}{b-a}\int_a^b f(x)dx$$

For some number $\bar{x}$ in $[a,b]$.

Theorem 2：The Fundamental Theorem of Calculus

Support that the function f is continuous on the closed interval $[a,b]$.

Part 1 If the function F is defined on $[a,b]$ by $F(x)=\int_a^x f(t)dt$

Then F is an antiderivative of f. That is, $F'(x)=f(x)$ for x in (a,b).

Part 2 If G is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx=[G(x)]_a^b=G(b)-G(a)$$

§5-7 Integration by Substitution

Theorem 1：Definite Integral by Substitution

Support that the function g has a continuous derivative on $[a,b]$ and that f is continuous on the set $g([a,b])$. Let $u=g(x)$. Then

$$\int_a^b f(g(x))\cdot g'(x)=\int_{g(a)}^{g(b)} f(u)du$$

§5-8 平面區域的積分

The Area Between two Curves

The area Between Two Curves

§5-9 Numerical Integration