Infinite Series
無窮級數

§11-1

$a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots$

§11-2 Infinite Sequences（無限序列）

Limit of Sequence

We say that the sequence $\{S_n\}$ converges to the real number L, or has the limit L, and we write

$\lim_{n\to\infty} S_n = L$

Provided that S_n can be made as close to L as we please merely by choosing n to be sufficiently large. That is, given any number $\varepsilon > 0$, there exist an integer N such that

$|S_n - L| < \varepsilon$ for all $n \ge N$

If the sequence $\{S_n\}$ does not converge, then we say that $\{S_n\}$ diverges.

序列收斂（converge）與發散（diverges）

§11-3 收斂無窮級數

無窮級數（infinite series）：$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots$

a_n 稱爲此級數的第 n 項（nth term）

前 n 項和（nth partial sum）S_n ：$S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$

部分和的序列（sequence of partial sums）：$S_1, S_2, S_3, \ldots, S_n, \ldots$

The sum of an Infinite Series

We say that the infinite series $\sum_{n=1}^{\infty} a_n$ converges (or is convergent)

With sum S provided that the limit of its sequence of partial sums,

$S = \lim_{n\to\infty} S_n$

Exist (and is finite). Otherwise we say that the series diverges (or is divergent). If a series diverges, then it has no sum.

$$S=\sum_{n=1}^{\infty}a_n=\lim_{N\to\infty}\sum_{n=1}^{N}a_n$$

【補充】等差數列與等差級數（A.P.）：

一數列爲 a_1 、 a_2 、 a_3 、 a_4 、…….. a_n

其中 $a_2-a_1=a_3-a_2=.........=a_n-a_{n-1}=d$

就稱此數列爲等差數列，而相鄰兩項的差 d 稱爲“公差”。

第 n 項的值：一等差數列，首項爲 a ，公差爲 d ，第 n 項爲 a_n

則 $a_n=a+(n-1)d$

$a_n=a_k+(n-k)d$

已知 a 、 b 、 c 爲等差數列，其中 $b=\frac{a+c}{2}$ ，則稱 b 爲等差中項

等差級數的和： $S_n=\frac{(a_1+a_n)n}{2}=\frac{n}{2}[2a_1+(n-1)d]$

$$\sum_{n=1}^{n}a_n=a_1+a_2+a_3+......+a_n$$
$$=a+[a+d]+[a+2d]+[a+3d]+......+[a+(n-1)d]$$
$$=an+[1+d+2d+3d+......(n-1)d]$$
$$=an+\frac{1+(n-1)d}{2}\times n=\frac{2a+d}{2}\times n$$

Geometric Series

The series $\sum_{n=0}^{\infty}a_n$ is said to be a geometric series if each term after the first is a fixed multiple of the term immediately before it. That is, there is a number r, call the ratio of the series, such that

$a_{n+1}=ra_n$ for all $n\ge 0$

$$a_0+ra_0+r^2a_0+r^3a_0+...=\sum_{n=0}^{\infty}r^na_0$$

$$S_n=a_0(1+r+r^2+r^3+...+r^n)$$

【補充】等比數列與等比級數（G.P.）：

一數列爲 a_1 、 a_2 、 a_3 、 a_4 、…….. a_n

其中 $\frac{a_2}{a_1}=\frac{a_3}{a_2}=\frac{a_4}{a_3}=......=\frac{a_n}{a_{n-1}}=r$

就稱此數列爲等比數列，而相鄰兩項的商數 r 就稱爲“公比”。

第 n 項的值：一等比數列，首項爲 a ，公比爲 r ，第 n 項爲 a_n

則 $a_n=ar^{n-1}$

$a_n = a_k r^{n-k}$

等比級數的和：$S_n = \dfrac{a_1(1-r^n)}{1-r}$（但 $r \neq 1$）

$$\sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \dots\dots + a_n$$
$$= a + ar + ar^2 + ar^3 + \dots\dots + ar^{n-1}$$
$$= a(1 + r + r^2 + r^3 + \dots\dots + r^{n-1})$$
$$= a\frac{1-r^n}{1-r}$$

$$1 + r + r^2 + r^3 + \dots\dots + r^{n-1} = \frac{r^n - 1}{r - 1}$$

<證明>：假設 $A = 1 + r + r^2 + r^3 + \dots\dots + r^{n-1}$

$rA = r + r^2 + r^3 + \dots\dots + r^n$

兩式相減 $A - rA = 1 - r^n$

$\therefore \quad A = \dfrac{1-r^n}{1-r} \dots \#$

Theorem 4
The harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

【補充】調和數列與調和級數（H.P.）：

一數列為 $\dfrac{1}{a}$、$\dfrac{1}{a+d}$、$\dfrac{1}{a+2d}$、…、$\dfrac{1}{a+(n-1)d}$、…其倒數為等差數列，稱為調和數列。

已知 a、b、c 為調和數列，可知 $\dfrac{1}{a}$、$\dfrac{1}{b}$、$\dfrac{1}{c}$ 為等差數列，其中 $b = \dfrac{2ac}{a+c}$，為調和中項。

§11-4 Taylor Series（泰勒級數）、Taylor Polynomials（泰勒多項式）

Theorem1：The nth-Degree Taylor Polynomial
Support that the first n derivatives of the function $f(x)$ exist at $x = a$. Let $P_n(x)$ be the n-th degree polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then the values of $P_n(x)$ and its first n derivatives agree, at $x = a$, with the value of f and its first n derivatives there.

$a = 0$，泰勒多項式可寫成 $P_n(x) = f(0) + f'(0)\cdot x + \dfrac{f''(0)}{2!}x^2 + \ldots + \dfrac{f^{(n)}(0)}{n!}x^n$

<例題>： $f(x) = e^x$ at $a = 0$

$f^{(k)}(x) = e^x$ for all $k \ge 0$

$f^{(k)}(0) = 1$ for all $k \ge 0$

因此 $P_n(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}$

<討論>：討論前幾項泰勒多項式

$P_0(x) = 1$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \dfrac{1}{2}x^2$

$P_3(x) = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3$

$P_4(x) = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$

$P_5(x) = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4 + \dfrac{1}{120}x^5$

當 $x = 0.1$ 時，

n	$P_n(x)$	e^x	$e^x - P_n(x)$
0	1.00000	1.10517	0.10517
1	1.10000	1.10517	0.00517
2	1.10500	1.10517	0.00017
3	1.10517	1.10517	0.00000
4	1.10517	1.10517	0.00000

當 $x = 0.5$ 時，

n	$P_n(x)$	e^x	$e^x - P_n(x)$
0	1.00000	1.64872	0.64872
1	1.50000	1.64872	0.14872
2	1.62500	1.64872	0.02372
3	1.64583	1.64872	0.00289
4	1.64844	1.64872	0.00028

最接近 $f(x)$ 的趨近值 $P_n(x)$，和原本的 $f(x)$ 會有一個差異值

$R_n(x) = f(x) - P_n(x)$

➲ $f(x) = P_n(x) + R_n(x)$

$R_n(x)$ 稱爲 n-th degree remainder for $f(x)$ at $a=0$ 。
Remainder（餘數）
其實它就是 $f(x)$ 與趨近值 $P_n(x)$ 的誤差值。

Theorem 2：Taylor's Formula

Support that the (n+1)th derivatives of the function f exist on an interval containing the points a and b. Then

$$f(b)=f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\frac{f^{(3)}(a)}{3!}(b-a)^3+\ldots$$
$$\ldots+\frac{f^{(n)}(a)}{n!}(b-a)^n+\frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

For some number z between a and b.

The n-th degree Taylor formula with remainder at $x=a$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\ldots$$
$$+\frac{f^{(n)}(a)}{n!}(x-a)^n+\underbrace{\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}}_{=R_n(x)}$$

Where z is some number between a and x. Thus the n-th degree remainder term is

$$R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

<例題>： $f(x)=e^x$ then $f^{(k)}(x)=e^x$ for all $k\ge 0$

Taylor formula $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$

At $a=0$

$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\frac{e^z x^{n+1}}{(n+1)!}$ for some z between 0 and x.

Remainder term $R_n(x)$ 要滿足

$$0<|R_n(x)|<\frac{|x|^{n+1}}{(n+1)!} \quad \text{if } x<0$$

$$0<|R_n(x)|<\frac{e^x x^{n+1}}{(n+1)!} \quad \text{if } x>0$$

➲ $\lim_{n\to\infty}\frac{x^n}{n!}=0$

➲ $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$

$x=1$ 代入

➲ $e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$

➲ $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{10!}\approx 2.71828\ 18$

$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{20!}\approx 2.71828\ 18284\ 59045\ 235$

<例題>： $f(x)=\cos x$ at $a=0$

$f(x)=\cos x$ $\qquad f'(x)=-\sin x$

$f''(x)=-\cos x$ $\qquad f^{(3)}(x)=\sin x$

$f^{(4)}(x)=\cos x$ $\qquad f^{(5)}(x)=-\sin x$

$\vdots \qquad \vdots$

$f^{(2n)}(x)=(-1)^n\cos x$ $\qquad f^{(2n+1)}(x)=(-1)^{n+1}\sin x$

➲ $f^{(2n)}(0)=(-1)^n$ $\qquad f^{(2n+1)}(0)=0$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-...+(-1)^n\frac{x^{2n}}{(2n)!}+(-1)^{n+1}\frac{\cos z}{(2n+2)!}x^{2n+2}$$

Where z is between 0 and x

因爲 $|\cos z|\le 1$ for all z

$\therefore$ 當 $n\to\infty$ ， $R_n(x)\to 0$ for all x

➲ $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...$

<例題>： $f(x)=\sin x$ at $a=0$

➲ $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=1-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$

The case $a=0$ of Taylor's series is called the Maclaurin series of the function $f(x)$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+...$$

The three Maclaurin series $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=1-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$$

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The harmonic series is the case $p=1$ of the p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

Where the p-series converges or diverges depends on the value of p.

$p=2$ ➲ $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ converges

$p=\frac{1}{2}$ ➲ $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$ diverges

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【再補】

§11-7 Alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+a_5-\ldots$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$$

Theorem 1 Alternating Series Test

If $a_n>a_{n+1}>0$ for all n and $\lim_{n\to\infty}a_n=0$ then the alternating series in

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+a_5-\ldots$ converges.

§11-8 Power Series

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

Convergence of power series

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$u_n=a_nx^n$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \rho(x)$$

<例題>：$\sum_{n=0}^{\infty} n^n x^n$

<解答>：$u_n = n^n x^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right|$$

$$= \lim_{n\to\infty}(n+1)\left(1+\frac{1}{n}\right)^n |x|$$

$$= +\infty$$

For all $x \neq 0$, because $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^n}{n!} + \ldots$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$
